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The University of Alberta

An Introduction to Thermo Field Dynamics
-- Equilibrium and Non-Equilibrium
(Including Comparisons with Other Thermal Theories)

by

© James Fred Pradko

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF ...Masters of Science.....

DEPARTMENT OF Physics.....

EDMONTON, ALBERTA

(SPRING)(1986)

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To whom it may concern;

This letter is to serve a formal verification that we, Dr. T. Arimitsu and Dr. H. Umezawa agree to allow Mr. J. Pradko to contain within his thesis any amount, or expression of, our paper;

"Generating Functional Methods in Non-Equilbrium Thermo Field Dynamics" by T. Arimitsu, J. Pradko and H. Umezawa.

Currently this paper is a preprint of the University of Alberta and has been submitted for publication but has not, as yet, been accepted for publication.

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled AN INTRODUCTION TO THERMO FIELD DYNAMICS -- EQUILIBRIUM AND NON-EQUILIBRIUM (INCLUDING COMPARISONS WITH OTHER THERMAL THEORIES) submitted by James Pradko in partial fulfilment of the requirements for the degree of Masters of Science in Physics.

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(Supervisor)

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Date: September 24, 1985

Dedication

"To live in the company of Men-at-their-best is the finest thing possible ..."

Confucius

This thesis is dedicated to the three people-at-their-best who have helped to shape my life:

My Mother, who supports me with a love that never fails and spands every distance.

My Father, who's trust and loving support have given me the confidence to accomplish that which I was unsure of; and

Dr. Umezawa, who has a depth to his understanding which is most rare, and has a way all his own of passing his understanding on, so that one is filled with inspiration and excitement.

Abstract

In this thesis quantum field theory is firstly presented with careful attention paid to its axioms, their meaning and implications. A generalization is made by introduction of the σ-commuting tilde field and an equilibrium finite temperature field theory is presented, namely Thermo Field Dynamics. A second generalization is made with the introduction of supervectors and superoperators allowing Thermo Field Dynamics to be further developed to handle non-equilibrium situations. Comparisons between Thermo field Dynamics and other thermal theories is also presented.

Preface

This thesis "An Introduction to Thermo Field Dynamics -- Equilibrium and Non-Equilibrium, (Including Comparisons With Other Thermal Theories)" is intended to be self contained. The first part "Quantum Field Theory Without Thermal Degrees of Freedom" presents to the reader the most basic aspects of Quantum Field Theory. Starting with the definition of, and need for a Hilbert space in Quantum Mechanics, many particle states are introduced. The first real difference between Quantum Mechanics and Quantum Field Theory is pointed out, namely that due to the fact that we have infinite degrees of freedom the set $\{|n\rangle\}$ does not form a countable basis and the proper choice from the many possible countable subsets of $\{|n\}\rangle$ must be made utilizing a self-consistent consideration. Once the Fock space is arrived at which has a countable basis set, the Fourier representation for this physical representation is introduced. The ordinary perturbative calculation procedure is briefly discussed for completeness and this prepares the way for a discussion of the dynamical map and its implication that the full Hamiltonian is weakly equal to the free Hamiltonian. With these realizations we again return to the subject of free fields in order to develop the relations (sum rules, etc.) needed to present the last three sections of this part. The L.S.Z. formulation is mentioned as we introduce the 'in' and 'out' fields in section 9. Section 10

contains a derivation of the reduction formulas and the L.S.Z. formula. Part I ends with a discussion of the two-point Green's functions of quantum field theory without thermal freedom, and closing remarks.

Part II of this work entitled "Finite Temperature Quantum Field Theory" starts off with the motivating realization that a thermal vacuum can be defined which yields the usual statistical mechanical expectation values. The thermal vacuum is defined and normalized in the next two sections. The thermal state condition, which is now known to be one of the most basic entities of thermo field dynamics (TFD) is derived for equilibrium situations. The tremendous physical significance of the thermal state condition is touched upon but is discussed more fully and demonstrated in part III. Following this the thermal vacuum as a pure quantum state is discussed and it is shown possible to include all phase information within this thermal ground state. At this point part I's material can be generalized to accommodate thermal degrees of freedom using the now well defined quantum field theory at finite temperature; TFD. These generalizations include; the dynamical map, Heisenberg fields, Kubo-Schwinger-Martin relation and the L.S.Z. formula.

The original title of this thesis did not contain the word "Non-Equilibrium" because at the time of its conception the ability of TFD to describe non-equilibrium situations was

not understood. Since then TFD for non-equilibrium situations has been well developed and it is very exciting to include an introduction to it in this thesis. This introduction makes up part III. Part III starts with a discussion of the thermal Liouville space and the super-operators and supervectors which define it. The different representations of this space are developed, namely the interaction and Heisenberg representation. The thermal state condition is derived in its more general (non- equilibrium) form and it is shown how this condition defines our quasi-particle operators. The section containing the seven axioms of TFD follows next. These axioms allow the entire development of a formalism which describes non-equilibrium systems: There is no mention of a reservoir in these axioms. The formalism thus far developed is then applied to a phase-invariant bilinear model which makes the proper use of the axioms very clear. This part ends with the expression and discussion of the generating functional for TFD in non-equilibrium situations, which is the author's original contribution. Throughout this final section comparisons are made to the generating functional formalism of Schwinger.

The material in the thesis thus far accomplishes a description of nature that has been sought after for many years. In the final part of this work appear four short comparisons between TFD and the older thermal theories, which

are not field theories and hence can not make use of the vast calculational methods of quantum field theory. The thermal theories discussed include the path ordering method, Schwinger's path ordering method, sub-dynamics and c^*-algebra.

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Luisa, Martin, Michelle, Nathalie, On, Paul, Raven, Rick, Sara and Suchira. Throughout the study and writing of this thesis I have received much support including financial, from the Dept. of Physics at The University of Alberta. I would like to thank this department, of which I am glad to be a member.

Lastly I would like to acknowledge and gratefully thank my advisor, Dr. H. Umezawa. For the time he has spent with me, I am thankful. For the thoughts he has conveyed to me, I am wiser. I thank him for allowing me to become a member of his energetic, productive and very active research group. Dr. Umezawa's vast scope of knowledge and quickness in recall constantly inspires me and I thank him for clarifying much of my previous understanding and for teaching me so very much; enabling this thesis to be written.

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PART I: QUANTUM FIELD THEORY WITHOUT THERMAL DEGREES OF FREEDOM

§0. Introduction

Although for some readers this section (or parts thereof) may be skipped, the purpose in its appearance is two fold: Firstly, it will make clear in an operational way the notation to be used throughout this work and secondly, it is designed to present and discuss field theory without thermal degrees of freedom by itself so that later we may concentrate specifically on how thermal effects influence these considerations. This section should then be read with the understanding that all of its contents will, in later sections, be generalized in order to accommodate thermal degrees of freedom.

§1. One Particle States

Quantum mechanics is built within the abstract notion of a linear vector space. A linear vector space is a set of elements, called vectors, which have defined over them two operations; addition and scalar multiplication. These operations are such that for arbitrary vectors ψ, φ, χ and c-numbers a and b; we have

$$\psi + \varphi = \varphi + \psi$$

$$\psi + (\varphi+\chi) = (\psi+\varphi) + \chi$$

There exists a vector 0 such that $\psi + 0 = \psi$

$$a(\psi+\varphi) = a\psi + a\varphi$$

$$(a+b)\psi = a\psi + b\psi$$

$$(ab)\psi = a(b\psi)$$

$$1\psi = \psi$$

$$0\psi = 0 \ . \qquad (1.1)$$

We may define an inner product such that

$$(\psi,\varphi+\chi) = (\psi,\varphi) + (\psi,\chi)$$

$$(\psi,a\varphi) = a(\psi,\varphi)$$

$$(\psi,\varphi) = (\varphi,\psi)^*$$

$$(\psi,\psi) \geq 0 \text{ with equality only for } \psi=0 \ . \qquad (1.2)$$

If we have an infinite dimensional linear vector space with an inner product we say that it is 'complete' if for a spanning set of orthonormal vectors $\{\varphi_i(x)\}$, vector $\psi(x)$ given by

$$\psi(x) = \sum_{i=1}^{\infty} a_i \varphi_i(x) \qquad (1.3)$$

where

$$\| \psi_n(x) - \psi_m(x) \| = \| \sum_{k=1}^{n} a_i \varphi_i(x) - \sum_{k=1}^{m} a_i \varphi_i(x) \| \to 0 \quad \text{as } n,m \to \infty \qquad (1.4)$$

is also an element of the space. Equation (1.4) is the condition defining a Cauchy sequence. ψ may be called a limit vector. Without proof we state that an orthonormal set of vectors is countable[1]. A complete linear vector space with an inner product is called a Hilbert space. A separable Hilbert space is one which has a countable or equivalently an orthonormal basis. In quantum mechanics we take our physical one particle states to be represented by vector elements of a separable Hilbert space because each of the above mentioned conditions on the elements of a Hilbert space ($\mathcal{H}$) are necessary and sufficient for a successful abstraction of a physical particle's state into a mathematical formalism.

The addition operation is an abstraction of the physical reality that particles obey the superposition principle. The complex scalar multiplication operation is an abstraction of the physical reality that one particle states carry different phases and amplitudes and the inner product is an abstraction of the physical reality that the observation of a certain state has a probability proportional to the degree of similarity between the actual and observed

state. This correspondence between vectors $\psi(x)$ and a probability demands that the inner product be normalizable

$$\int d^3x(\psi(x),\bar{\psi}(x)) < \infty \ . \qquad (1.5)$$

Hence by adjusting the amplitude of $\psi(x)$ using scalar multiplication we can have the adjusted $\psi'(x)$ such that

$$\int d^3x(\psi(x)',\bar{\psi}(x)') = 1 \quad \text{for all } \psi(x)' \in \mathcal{H} \ . \qquad (1.6)$$

Realize that equation (1.5) precludes plane waves as a possible basis for our separable Hilbert space. The abstract condition that our Hilbert space be separable and hence have a countable basis is a requirement reflecting the physical reality that any physically realizable state can be expressed in terms of a suitable chosen observable set. As will be discussed, the suitability of a physical basis is a very fundamental question.

Hence, if we would like to adopt some abstract formalism as the language with which we may speak about physical particles which obey the laws of quantum physics we must make sure that the above requirements are met by that formalism. The reason then for a separable Hilbert space is clear.

§2. Many Particle States

Instead of using the basis set $\{\varphi_i(x)\}$ of equation (1.3) and thus having to stipulate the set $\{a_i\}$ for every particle state, we now introduce the occupation number representation. Let us consider a many particle state: It can be completely described by stating the number of particles found in each of the basis states, i.e., the number of particles found to have a wavefunction equal to each element of $\{\varphi_i(x)\}$. Such strings of numbers are called elements of the second-quantized representation:

$$\psi(x) = |n_1,n_2,\ldots\rangle \ . \tag{2.1}$$

Equation (1.1) are easily understood for such state vectors and introducing a conjugate vector

$$\langle n_1,n_2,\ldots| = |n_1,n_2,\ldots\rangle^\dagger \tag{2.2}$$

and the annihilation and creation operators coming up in section §3 we can understand equations (1.2).

We do however have a problem, in that the basis set $\{|n_1,n_2,\ldots\rangle\}$ is not countable and hence upon this basis we cannot build a separable Hilbert space. This non-countability of the set $\{|n_1,n_2,\ldots\rangle\}$ which will be denoted by $\{|n\rangle\}$ from now on, can be seen when one realizes that there is a one to one correspondence between the numbers which cover the interval (0,1) and the set $\{|n\rangle\}$ for fermions and for bosons if one writes their occupation number with binomial numbers. Since the former is non-countable we have

that latter is also non-countable. In order to continue then in the preparation of an abstract structure capable of giving a faithful expression of reality we need only to select out of the set $\{|n\rangle\}$ a subset which is both countable and complete enough to be able to express any physically realizable state. Each separable Hilbert space built upon each of the many such countable subsets will be disjoint or unitarily inequivalent and thus unitary transformations from one such separable Hilbert space to another are not possible.

§3. A Suitable Subset of {|n>}

We choose the subset '[0]' from {|n>} which is to be the basis of our separable Hilbert space, it is given by

$$[0] \equiv \{|n_1,n_2,\ldots\rangle;\ \sum_i n_i = \text{finite}\}\ , \qquad (3.1)$$

this basis is countable[2], it also contains the convenient cyclic vector $|0\rangle=|0,0,\ldots\rangle$ upon which the rest of the space may be constructed. To this end we introduce operators α_i and $\alpha_i^\dagger$ which for bosons are defined by

$$\alpha_i|n_i,\ldots n_i\ldots\rangle \equiv n_i^{1/2}|n_1,\ldots n_i-1\ldots\rangle \qquad (3.2)$$

$$\alpha_i^\dagger|n_1,\ldots n_i\ldots\rangle \equiv (n_i+1)^{1/2}|n_1,\ldots n_i+1\ldots\rangle. \qquad (3.3)$$

Observe that, with $[a,b]_\pm = ab \pm ba$, we have

$$[\alpha_i,\alpha_j^\dagger]_- = \delta_{ij} \qquad (3.4)$$

$$[\alpha_i,\alpha_j]_- = [\alpha_i^\dagger,\alpha_j^\dagger]_- = 0\ . \qquad (3.5)$$

For fermions we define α_i and $\alpha_j^\dagger$ as

$$\alpha_i|n_1,\ldots n_i\ldots\rangle \equiv (-1)^{\sum_{j<i} n_j}|n_1,\ldots n_i-1\ldots\rangle \qquad (3.6)$$

for $n_i=1$, otherwise zero

$$\alpha_i^\dagger|n_1,\ldots n_i\ldots\rangle \equiv (-1)^{\sum_{j<i} n_j}|n_1,\ldots n_i+1\ldots\rangle \qquad (3.7)$$

for $n_i=0$, otherwise zero.

Observe that

$$[\alpha_i, \alpha_j^\dagger]_+ = \delta_{ij} \tag{3.8}$$

$$[\alpha_i, \alpha_j]_+ = [\alpha_i^\dagger, \alpha_j^\dagger]_+ = 0 \ . \tag{3.9}$$

Now any vector $|n_1, n_2, \ldots\rangle$ may be written

$$|n_1, n_2, \ldots\rangle = \prod_i (n_i!)^{-1/2} (\alpha_i^\dagger)^{n_i} |0\rangle \tag{3.10}$$

where for fermions 'n_i' takes on only the values 1 and 0 and the order of the developed product is preserved. Clearly $|0\rangle$ is a cyclic vector. Using equations (3.2) through (3.9) we have

$$\langle n_1', n_2', \ldots | n_1, n_2, \ldots\rangle = \prod_i \delta_{n_i' n_i} \tag{3.11}$$

and hence our set [0] is an orthonormal set. Consider now the set of vectors $\mathcal{H}$ '[α]' defined by

$$[\alpha] \equiv \{\psi = \sum_i^\infty a_i |n\rangle_i \ ; \ \sum_i |a_i|^2 = \text{finite}; \ |n\rangle_i \in \{[0]\}\}. \tag{3.12}$$

Such a Hilbert space is called a Fock space. Equation (3.12) is a Cauchy sequence and hence the limit vector ψ can be approximated to any degree of accuracy by

$$\psi' = \sum_i^N a_i |n\rangle_i \ ; \quad N = \text{finite}. \tag{3.13}$$

The set formed by all linear combinations of the vectors appearing in Equation (3.13) is the set of all vectors formed

by cyclic action of $\alpha_i^\dagger$ on $|0\rangle$, i.e., equation (3.10). This set is said to be dense in $\mathcal{H}[\alpha]$.[3] We are now at the point we have hoped for: We have a set $\mathcal{H}[\alpha]$, our Fock space, which by arguments given in section §2, is a Hilbert space. From equation (3.11) we see it is a separable Hilbert space and we have a way to build to any degree of accuracy any element of it by cyclic action of $\alpha_i^\dagger$ on $|0\rangle$. It is within this mathematical space then that we will continue our discussion of many body physics.

One notices a strange thing, namely that implicit to our definition of the Fock space $\mathcal{H}[\alpha]$ is the need for a specification of $\{\alpha_i^\dagger\}$. Each different representation of $\alpha_i^\dagger$ yields a different unitarily inequivalent Fock space. We will discuss this thoroughly in section 5 and we will see that the specific choice of Fock space comes from a self-consistent consideration.

§4. Fourier Representation

Thus far the spacial distribution of the wave functions $\varphi_i(x)$ has only been said to be normalizable, and hence a wave packet. A single particle state with an arbitrary spacial distribution different from that of any element of the basis set [0] may be written

$$f(x) = \sum_i c_i \varphi_i(x) \tag{4.1}$$

$$g(x) = \sum_i d_i \varphi_i(x) \tag{4.2}$$

We define

$$a_f^\dagger = \sum_i c_i a_i^\dagger \tag{4.3}$$

with Hermitian conjugate

$$a_f = \sum_i c_i^* a_i \tag{4.4}$$

Observe that

$$\begin{aligned} [a_f, a_g^\dagger]_\pm &= \sum_{ij} c_i^* [a_i, a_j^\dagger]_\pm d_j \\ &= \sum_i c_i^* d_i \\ &\equiv (f,g) \end{aligned} \tag{4.5}$$

$$[a_f, a_g]_\pm = [a_f^\dagger, a_g^\dagger]_\pm = 0 \tag{4.6}$$

We now choose to define our annihilation and creation operators (in momentum space) as

$$[a(k), a(\ell)^\dagger]_\pm = \delta(k-\ell) \tag{4.7}$$

$$[a(k), a(\ell)]_\pm = [a(k)^\dagger, a(\ell)^\dagger]_\pm = 0 \tag{4.8}$$

where the δ-function implies that these equations are to be

understood in the sense of distribution theory. In other words with suitable test function, which f(k) and g(ℓ) of equation (4.6) are. We have for equation (4.7)

$$\int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3\ell}{(2\pi)^{3/2}} f(k^*)g(\ell)[a(k),a(\ell)^\dagger]_\pm =$$

$$\int \frac{d^3k}{(2\pi)^3} f^*(k)g(k) = (f,g) \ . \tag{4.9}$$

It makes sense then to identify

$$a_f^\dagger = \int \frac{d^3k}{(2\pi)^{3/2}} f(k)a^\dagger(k) \tag{4.10}$$

which is consistent with equations (4.5) and (4.6). If we consider a single particle element of the basis set [0], we have from equation (4.10)

$$a_i^\dagger(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \varphi_i(k)a^\dagger(k) \tag{4.11}$$

where $\varphi_i(k)$ is the Fourier transform of $\varphi_i(x)$. Equation (4.11) agrees with our previous understanding that $a_i^\dagger$ is the creation operator for a particle with spacial distribution $\varphi_i(x)$. It is important to realize that vectors formed by cyclic action of $a(k)$ and $a(k)^\dagger$ on $|0\rangle$ are not elements of our Fock space for they have infinite norm,

$$\langle 0|a(k)a(k)^\dagger|0\rangle = \delta^3(0) \ . \tag{4.12}$$

Their utility is great however and from the correspondence given in equation (4.11) they play a critical role in the Fourier representation of $a_i^\dagger$.

§5. The Fock Space of Physical Particles

As mentioned earlier each representation of $a_i^\dagger$ leads to a different (unitarily inequivalent) Fock space. We envision our Hilbert space as containing all observable states and hence we require that our Hilbert space be the Fock space in which physical particles appear in observation. In other words we choose $\mathcal{H}[a]$ as our space where $a_i^\dagger$ is the creation operator for physically observed quanta or particle. $|0\rangle$ is then our physical vacuum. We call this representation of the Fock space the physical particle representation.

One usually observes an individual excitation which can be described as a free particle. Therefore, all observed particles are contained in the physical Fock space which contains all state vectors for free particle states even if there are interactions present.

If a single physical particle has momentum $\vec{k}$ and energy $\omega(\vec{k})$ then the total energy of an interacting or noninteracting closed many body system must be just the sum of each free particle's energy. Our free Hamiltonian then gives

$$H_o\, a^\dagger(k_1)\ldots a^\dagger(k_n)|0\rangle = \Big(\sum_{i=1}^{n} \omega(k_i)\Big)\, a^\dagger(k_1)\ldots a^\dagger(k_n)|0\rangle \tag{5.1}$$

with $\hbar = 1$, or

$$[H_o,a^\dagger(k_1)]_-a^\dagger(k_2)\ldots a^\dagger(k_n)|0\rangle = \omega(k_1)a^\dagger(k_1)\ldots a^\dagger(k_n)|0\rangle. \tag{5.2}$$

This means

$$[H_o,a^\dagger(k)]_- = \omega(k)a^\dagger(k) \tag{5.3}$$

and

$$[H_o,a(k)]_- = -\omega(k)a(k). \tag{5.4}$$

H_o then has the simple form

$$H_o = \int d^3k\ \omega(k)a^\dagger(k)a(k). \tag{5.5}$$

For wave packet states with special distribution f(x), we have in momentum space,

$$[H_o,a_f^\dagger] = \int \frac{d^3k}{(2\pi)^{3/2}}\,\omega(k)f(k)a^\dagger(k). \tag{5.6}$$

Hence

$$H_o a_F^\dagger|0\rangle = \int \frac{d^3k}{(2\pi)^{3/2}}\,\omega(k)f(k)a^\dagger(k)|0\rangle. \tag{5.7}$$

Since observable states must be wave packet states we see that observable states are not pure eigenstates of the free Hamiltonian. Of course careful preparation of our wave function can yield one arbitrarily close to a plane wave. The arguments of this section thus far can also be carried out for momentum operator P

$$P = \int d^3k\ \vec{k}\ a^\dagger(k)a(k) \tag{5.8}$$

The free Hamiltonian can not describe an interchange of energy characteristic of interaction among particles, although it describe the system's total energy. We introduce dynamics into our system through dynamical canonical

Heisenberg field operators '$\psi(x)$'. The spacial and temporal behavior of the Heisenberg fields is governed by the Heisenberg equation; $i\frac{\partial \psi}{\partial t} = [\psi, H]$ where H is the 'full Hamiltonian' and hence is the operator which generates time translation in the presence of interaction. We will show in section 7 that

$$\langle a|H|b\rangle = \langle a|H_o|b\rangle + c\langle a|b\rangle, \qquad (5.9)$$

where $\langle a|, \langle b|$ are elements of physical Fock space 'c' is a c-number.

One should not be surprised at this relation; Equation (5.9) is not a statement that $H - H_o \equiv H_{int}$ is zero but a statement that the reaction does not consume energy. All the energy must be vested in the physical particles. In other words the total energy of a system is the sum of the free particles' energies.

Here then is the so aptly named "dual language of quantum field theory"[4] and the heart of the problem in quantum field theory. Given a Hamiltonian H, a function of Heisenberg fields (the first language) we seek to realize these fields in our physical Fock space (the second language), for this realization reflects the discussion of the previous paragraph. This "dynamical translation" from Heisenberg to physical fields can only be determined in a self-consistent calculation. For knowledge of such a mapping requires knowledge of the physical Fock space which in turn

requires knowledge of all observable states, and these are entirely sensitive to the dynamics of the system as described by the full Hamiltonian which, to start with, is a function of Heisenberg fields and not physical fields. Here by physical fields we mean an operator which is a linear superposition of the physical annhiliation and creation operators and with a free field equation describe the dynamics of the free physical particles; these fields will be denoted by $\varphi(x)$ as opposed to the Heisenberg fields denoted by $\psi(x)$. The Heisenberg fields '$\psi(x)$' may be expressed in terms of Heisenberg operators $a(x)$ and $a(x)^\dagger$ (analogous to $\varphi(x) = \varphi(\alpha(x)\alpha(x)^\dagger)$). The operators $a(x)$ and $a(x)^\dagger$ obey the same commutation relations as $\alpha(x)$ and $\alpha(x)^\dagger$ but they do not create or annihilate the physical particles and hence $a(x)$ does not annihilate the vacuum. To make clear the just described self-consistent nature of our problem imagine we have a Hamiltonian which contains two Heisenberg fields. We could guess that we need two physical particle fields which have statistics corresponding to the Heisenberg canonical relations which must also be given. We write out the 'dynamical map' which describes the many <u>physical</u> situations (collisions) which give rise to the existence of the Heisenberg field at a certain time and space. We now check the canonical relations and equation (5.9) working in the Fock space defined by the two assumed physical particles. If these relations are found to be satisfied we must have the

correct physical fields. If they are not satisfied it tells us that the dynamics in the full Hamiltonian will yield physically observed states which are not vectors in the Fock space we tried, ie. our Fock space is not the physical Fock space, it is not complete enough to describe reality. An example of this would be if the Hamiltonian we are using had an interaction term capable of producing a composite particle, hence our guess of two physically observable particles was in error for there are actually three.[5]

The time dependence of physical particle operators $a(k)$ and $a^{\dagger}(k)$ is generated by the free Hamiltonian H_o as

$$a(k,t) = e^{iH_o t} a(k) e^{-iH_o t} = a(k)\exp(-i\omega(k)t) \quad (5.10)$$

and its hermitian conjugate, which follows from equation (5.1). The time dependence of Heisenberg operators $a(k)$ and $a^{\dagger}(k)$ is generated by the full Hamiltonian H, through the equation

$$i \frac{d}{dt} a(k,t) = [a(k,t),H] \quad (5.11)$$

or equivalently

$$a(k,t) = e^{iHt} a(k) e^{-iHt} \quad (5.12)$$

The dynamical map then takes a general form

$$a(k,t) \triangleq F[a(k,t)], \quad (5.13)$$

for a one Heisenberg, one physical field situation. The symbol $\triangleq$ means it is a weak relation or an equality of matrix elements with respect to the physical Fock space and not an equality of the operators alone. Again the best one can do

is guess the elements $\alpha(k)$ and formulate a dynamical map as in equation (5.13), then one checks to see if all of the canonical relations and equation (5.9) which the left hand side of equation (5.13) satisfies, are indeed satisfied by the right hand side. Clearly equations (5.10) and (5.12) and hence (5.13) hold for functions of $a(k,t)$ and $\alpha(k,t)$ ie. $\psi(x,t)$ and $\varphi(x,t)$ because of equation (5.9).

§6. Perturbative Calculation

It is very useful to separate the time evolution of the states and the time evolution of operators such that they are generated by H_o and $H - H_o$ respectively. Such a representation is called the "interaction representation.[6] Given a Heisenberg operator $\psi(x)$ the reader will be familiar with the expression

$$\psi(x) = U^{-1}(t,t_o)\psi_{int}(x)U(t,t_o) \tag{6.1}$$

where

$$U(t,t_o) = 1 + (-i)\int_{t_o}^{t} dt_1 H_{int}(t_1) + (-i)^2\int_{t_o}^{t} dt_1 H_{int}(t_1) \times \int_{t_o}^{t_1} dt_2\, H_{int}(t_2) + \ldots \tag{6.2}$$

The Dyson chronological operator 'P', which orders operators in order of increasing time from right to left, can be tailored so that it accounts for fermion exchanges and is denoted by 'T'. Equation (6.2) then becomes

$$U(t,t_o) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_o}^{t} dt_1 \int_{t_o}^{t} dt_2 \int_{t_o}^{t} dt_n$$

$$T(H_{int}(t_1)\ldots H_{int}(t_n)) \tag{6.3}$$

$$= \exp\left(-i\int_{t_o}^{t} H_{int}(t_1)dt_1\right) \tag{6.4}$$

Hence

$$\psi(x) = \exp\left(i \int_{t_o}^{t} H_{int}(t_1)dt_1\right)\psi_{int}(x) \exp\left(-i \int_{t_o}^{t} H_{int}(t_1)dt_1\right). \tag{6.5}$$

Such an equation is suspect as it stands, for does the

infinite series $U(t,t_o)$ converge? The smallness of terms is meaningless in an operator sequence and hence one examines convergence with respect to matrix elements. Intuitively $\langle a|U(t,t_o)|b\rangle$ seems to converge because the increasing interaction complexity of each sucessive term supports the view that they are less likely: we shall say no more about the convergence of $U(t,t_o)$, except that there is no evidence which suggests that $U(t,t)$ converges uniformly. In the limits, $t \to -\infty$ and $t \to +\infty$ one should be aware that convergence of each term (integral) is guaranteed by the fact that we are taking matrix elements with respect to normalizable and hence wave packet states. The adiabatic trick[8] which artifically induces convergence should be viewed as taking the weak relation

$$\psi(x) \doteq U^{-1}(t,t_o)\psi_{int}(x)U(t,t_o) \tag{6.6}$$

into the strong relation

$$\psi(x) = U^{-1}(t,t_o)_\varepsilon \, \psi_{int}(x)U(t,t_o)_\varepsilon \tag{6.7}$$

where the subscript ε implies $H_{int}(t) \to H_{int}(t)\exp(-\varepsilon|t|)$ for small ε.

§7. The Dynamical Map and $H = H_o$

As stated earlier the Heisenberg fields $\psi(x)$, which carry the dynamics of a system, must be expressible in terms of physical fields because observation yields only physical particles. Let us assume that $\varphi^o(x)$ is the free physical field associated with Heisenberg field $\psi(x)$. We write (following closely the discussion in reference (2)) equation (5.13) in a more general form

$$\psi(x) = \psi(x,\varphi^o) \tag{7.1}$$

which in turn may be written as

$$\psi(x) = a + z^{1/2}\varphi^o(x) + \int d^4y \int d^4z\ F(x-y,x-z)N(\varphi^o(y)\varphi^o(z)) + \ldots \tag{7.2}$$

where N is the normal ordering operator used so that no contractions appear on our dynamical map.

$z^{1/2}$ is a renormalization factor, and 'a' is a c-number (equal to zero for fermion fields) and the time integration covers the full range of dynamics $(-\infty,\infty)$.

Now regarding $\varphi^o(x)$ and $\psi(x)$ as being real we have

$$H_o = \int d^3k\ \omega(k)a^\dagger(k)a(k) \tag{5.5}$$

and

$$[H_o,\varphi^o(x)]_- = -i\frac{\partial}{\partial t}\varphi^o(x). \tag{7.3}$$

Equation (7.2) yields

$$i\langle a|[H_o,\psi(x)]_-|b\rangle = Z^{1/2}\frac{\partial}{\partial t}\langle a|\varphi^o(x)|b\rangle$$
$$+ \int d^4y\, d^4z\, F(x-y,x-z)\left(\frac{\partial}{\partial t_y}+\frac{\partial}{\partial t_z}\right)\langle a|N(\varphi^o(y)\varphi^o(z))|b\rangle$$
$$+ \ldots \qquad (7.4)$$

which upon partial integration, which is well defined due to the guaranteed convergence in time due to $|a\rangle$ being a wave packet, yields

$$\langle a|[H_o,\psi(x)]_-|b\rangle = -i\frac{\partial}{\partial t}\langle a|\psi(x)|b\rangle. \qquad (7.5)$$

The Heisenberg equation for $\psi(x)$ yields

$$\langle a|[H,\psi(x)]_-|b\rangle = -i\frac{\partial}{\partial t}\langle a|\psi(x)|b\rangle \qquad (7.6)$$

and hence from equations (7.4) and (7.5) we arrive at

$$\langle a|[H_o\psi(x)]_-|b\rangle = \langle a|[H,\psi(x)]|b\rangle. \qquad (7.7)$$

Hence for all vectors $\langle a|$ which are elements of our physical Fock space in which $\varphi^o(x)$ is realized we have equality between matrix elements of H and H_o.

§8. More on Free Fields

If $\lambda(\partial)$ of our free field equation

$$\lambda(\partial)\varphi^{o}(x,t) = 0, \tag{8.1}$$

is at maximum a second degree polynomial in '$\frac{\partial}{\partial t}$' then we may write

$$\lambda(\partial) = \lambda^{(o)}(\nabla) + i\,\lambda^{(1)}(\nabla)(\tfrac{\partial}{\partial t}) - \lambda^{(2)}(\nabla)(\tfrac{\partial}{\partial t})^2 \tag{8.2}$$

where $A(\nabla)$ is such that $A(\nabla)e^{ik\cdot x} = A(ik)e^{ik\cdot x}$, etc. We will, in this paper, understand $\lambda(\partial)$ to be hermitian, (if it is not one can easily apply a hermitization matrix[8] and make it so). We now define[7]

$$\overleftrightarrow{\Gamma} = \lambda^{(1)}(\nabla) - i\,\lambda^{(2)}(\nabla)(\tfrac{\partial}{\partial t} - \tfrac{\overleftarrow{\partial}}{\partial t}) = \lambda^{(1)}(\nabla) - i\lambda^{(2)}(\nabla)\,\frac{\overleftrightarrow{\partial}}{\partial t}\,. \tag{8.3}$$

observe

$$\frac{d}{dt}\int d^3x\; f(x,t)^{\dagger}\overleftrightarrow{\Gamma}g(x,t) = \int d^3x\; f(x_1,t)^{\dagger}(\lambda^{(1)}(\nabla)g(x_1t)$$

$$- i\int d^3x\; f(x,t)^{\dagger}((\tfrac{\partial}{\partial t})^2 - (\tfrac{\overleftarrow{\partial}}{\partial t})^2)\lambda^{(2)}(\nabla)g(x,t) \tag{8.4}$$

$$= -i\int d^3k\; f(x,t)^{\dagger}(\lambda(\partial) - \lambda(-\overleftarrow{\partial})|g(x,t) \tag{8.5}$$

$$= 0$$

when $f(x,t)$ and $g(x,t)$ are solutions to equation (8.1).

$$(f(x,t),g(x,t)) = \int d^3x\; f(x,t)^{\dagger}\overleftrightarrow{\Gamma}g(x,t) \tag{8.6}$$

then, is the inner product we introduced in section 1, apart from the fact that it is not positive definite (see equation 1.2). This problem will be addressed shortly.

The time development of the physical annihilation and creation operators was given as

$$a(k,t) = a(k)\exp(-i\omega(k)t) \tag{5.10}$$

$$a^{\dagger}(k,t) = a(k)^{\dagger}\exp(i\omega(k)t). \tag{8.7}$$

The spacial development is generated by $\hat{P}/\hbar$ as

$$a(k,x,t) = \exp(-i\,\frac{\hat{p}}{\hbar}\,x)a(k,t)\exp(i\,\frac{\hat{p}}{\hbar}\,x) \tag{8.8}$$

$$= a(k)\exp(i(kx - \omega(k)t)) \tag{8.9}$$

$$a^{\dagger}(k,x,t) = a^{\dagger}(k)\exp(-i(kx - \omega(k)t)) \tag{8.10}$$

The field equation (8.1) is said to be "type-one" if there exists a differential operator '$d(\partial)$' called a "divisor"[7] which satisfies

$$d(\partial)\lambda(\partial) = i\,\frac{\partial}{\partial t} - \varepsilon(\nabla) \tag{8.11}$$

and "type-two", if

$$d(\partial)\lambda(\partial) = -\left(\frac{\partial^2}{\partial t^2} + \omega^2(\nabla)\right). \tag{8.12}$$

Clearly if $\Delta_G(x,t)$ is the Green's function of equation (8.11) or equation (8.12) then $d(\partial)\Delta_G(x,t)$ is the Green's function of the field equation (8.1).

Solutions of the free field equation (assuming for now that it is of type-two) are of the form

$$u^r(k,x,t) = u^r(k)\exp(i(kx - \omega(k)t)) \tag{8.13}$$

$$v^r(k,x,t) = v^r(k)\exp(-i(kx - \omega(k)t)) \tag{8.14}$$

where r carries other degress of freedom, such as spin. From equation (8.4) we have

$$\int d\,x\; u^{r\dagger}(k,x,t)\overleftrightarrow{\partial_t}v^s(\ell,x,t) = 0 \tag{8.15}$$

$$\int d^3x\; v^{s\dagger}(k,x,t)\overleftrightarrow{\partial_t}u^r(\ell,x,t) = 0 \tag{8.16}$$

and we choose $u^r(k,x,t)$ and $v^r(k,x,t)$ such that

$$\int d^3x\ u^{r\dagger}(k,x,t)\overleftrightarrow{\partial_t}u^s(\ell,x,t) = \delta_{rs}\delta(k-\ell) \qquad (8.17)$$

$$\int d^3x\ v^{r\dagger}(k,x,t)\overleftrightarrow{\partial_t}v^s(\ell,x,t) = \rho\ \delta_{rs}(k-\ell). \qquad (8.18)$$

Equation (8.17) can be made positive because $\lambda(\vartheta)$ is defined only up to a sign, but once it is set equation (8.14) may be negative . $\rho = \pm 1$ is used in equation (8.18) to make it positive. This is how we solve the before mentioned problem of the non-positive definite inner product.

We are now in a position to specify the free physical field $\varphi^o(x,t)$ as

$$\varphi^o(x,t) = \sum_r \int d^3k(u^r(k)a^r(k)\exp(i(k\cdot\vec{x} - \omega(k)t))$$

$$+ v^r(k)\beta^{r\dagger}(k)\exp(-i(k\cdot\vec{x} - \omega(-k)t) \qquad (8.19)$$

for type-two fields, and

$$\varphi^o(x,t) = \sum_r \int d^3k(\theta(\varepsilon(k))u^r(k)a^r(k) + \theta(-\varepsilon(k))v^r(-k)$$

$$\beta^{r\dagger}(-k))\exp(i(k\cdot\vec{x} - \varepsilon(k)t)) \qquad (8.20)$$

for type-one fields.

Using equations (8.15) through (8.18) we can project out the annihilation or creation operators from $\varphi^o(x,t)$ as

$$a^r(k) = \int d^3x\ u^{r\dagger}(k,\vec{x},t)\overleftrightarrow{\partial_t}\varphi^o(\vec{x},t) \qquad (8.21)$$

$$\beta^r(k) = -\rho \int d^3x\ v^{r\dagger}(k,\vec{x},t)\overleftrightarrow{\partial_t}\varphi^o(\vec{x},t) \qquad (8.22)$$

for equation (8.15) and similarily for equation (8.16). It now behoves us to introduce the solution to the homogeneous Klein-Gordon equation, known as the Schwinger $\Delta^{\pm}$-function.

$$\Delta^{\pm}(x,t) = \mp i(2\pi)^{-3}\int \frac{d^3k}{2\omega(\pm k)} \exp(\pm i(k\cdot\vec{x} - \omega(\pm k)t)). \quad (8.23)$$

Observe

$$\Delta^{+}(-x) = -\Delta^{-}(x), \quad (8.24)$$

$$\Delta^{+}(x-y) + \Delta^{-}(x-y) = 0; \text{ at } t_x = t_y \quad (8.25)$$

$$\Delta^{+}(x-y) - \Delta^{-}(x-y) = -i(2\pi)^{-3}\int \frac{d^3k}{\omega(k)} \exp(ik\cdot(x-y) \text{ at } t_x = t_y \quad (8.26)$$

$$\delta t \frac{\partial \Delta^{\pm}(x)}{\partial t} = -\frac{1}{2}\delta(x)\delta(t) \quad (8.27)$$

$$(\frac{\partial^2}{\partial t^2} + \omega^2)\Delta^{\pm}(x) = 0. \quad (8.28)$$

Hence

$$\lambda(\partial)d(\partial)\Delta^{\pm}(x) = 0 \quad (8.29)$$

for type-two equations. We may write $\varphi^{o}(x)$ of equation (8.19) as

$$\varphi^{o\pm}(x-x') = d(\partial)\Delta^{\pm}(x-x')w, \quad (8.30)$$

where w is a constant vector of compatible dimensions. Our orthonormalized set of solutions $u^r(k,x)$ and $v^r(k,x)$ of equations (8.9) and (8.10) form a basis and allow the expansions

$$d(\partial)\Delta^{+}(x-x')w = \sum_r \int d^3k\ c_r^{+}(k,x')u^r(k,x) \quad (8.31)$$

$$d(\partial)\Delta^{-}(x-x')w = \sum_r \int d^3k\ c_r^{-}(k,x')v^r(k,x). \quad (8.32)$$

These equations may be inverted using the orthogonality condition, as

$$c_r^{+}(k,x') = \int d^3x\ u^{r\dagger}(k,x)\vec{\Gamma}^{+}d(\partial)\Delta^{+}(x-x')w \quad (8.33)$$

$$c_r^-(k,x') = -P \int d^3x\, v^{r\dagger}(k,x)\overleftrightarrow{\partial} d(\partial)\Delta^-(x-x')w. \quad (8.34)$$

Writing out $\Delta^+(x-x')$ explicitly we have

$$c_r^+(k,x') = -i \frac{1}{2w(k)} (u^{r\dagger}(k,x')\Gamma(k,k_o)d(k)w)_{k_o=\omega(k)}, \quad (8.35)$$

where $\Gamma(k,k_o) = \lambda^{(1)}(k) - 2k_o\lambda^{(2)}(k)$. (8.36)

But equation (8.2) and equation (8.36) give that

$$\Gamma(k,k_o) = \frac{\partial}{\partial k_o}\lambda(k);$$

which leads to

$$\Gamma(k,k_o)d(k) + \lambda(k)\frac{\partial}{\partial k_o}d(k) = 2k_o, \quad (8.36b)$$

where $\lambda(k)d(k) = k_o^2 - \omega^2(k)$ was used.

Hence we have

$$[u^r(k,x')^\dagger\Gamma(k,k_o)d(k)]_{k_o=\omega(k)} = 2\omega(k)u^{r\dagger}(k,x') \quad (8.37)$$

which yields

$$c_r^+(k,x') = -i\, u^{r\dagger}(k,x')w \quad (8.38a)$$

and similarily for Δ^-;

$$c_r^-(k,x') = i\,\rho\, v^{r\dagger}(k,x')w. \quad (8.38b)$$

With equation (8.31) we arrive at the "sum rules"

$$\sum_r \int d^3k\, u^r(k,x)u^{r\dagger}(k,x') = i\, d(\partial)\Delta^+(x-x') \quad (8.39)$$

$$\sum_r \int d^3k\, v^r(k,x)u^{r\dagger}(k,x') = -i\,\rho\, d(\partial)\Delta^-(x-x'). \quad (8.40)$$

For type-one equations these sum-rules are valid provided we make the correspondence

$$\varepsilon(k) = \omega(k) \quad (8.41)$$

$$\Delta^+(x) = -i(2\pi)^{-3} \int d^3k \exp(i(kx - \omega(k)t)) \qquad (8.42a)$$

which gives

$$\delta(t)\Delta^+(x) = -i\,\delta(x)\delta(t) \qquad (8.42b)$$

$$(i\frac{\partial}{\partial t} - \omega(\nabla)\Delta^+(x) = 0 \qquad (8.42c)$$

We replace equation (8.36b) with

$$\Gamma(k,k_o)d(k) + \lambda(k)\frac{\partial}{\partial k_o} d(k) = 1 \qquad (8.42d)$$

With equations (8.15) and (8.16) we have

$$[\varphi^o_k(x),\varphi^{o\dagger}_\ell(y)]_\pm = i\, d_{k,\ell}(\partial)(\Delta^+(x-y) + \rho\rho'\Delta^-(x-y)), \qquad (8.43)$$

where $\rho' = \begin{cases} 0 & \text{type-one with } \varepsilon(k)>0 \\ +1 & \text{type-two, boson} \\ -1 & \text{type-two, fermion} \end{cases}$ (8.44)

We now introduce the "condition of causality" as the guaranteed commutation at equal times of locally distinct observables (A(x)), and we see the observation of Pauli that

$$[A_i(x,t),A_j(y,t)]_- = 0 \quad \text{for all } x=y, \qquad (8.45)$$

implies

$$[\varphi^o_j(x,t),\varphi^{o\dagger}_k(y,t)]_\pm = 0 \quad \text{for all } x=y, \qquad (8.46)$$

and hence we must have

$$\rho\rho' = 1. \qquad (8.47)$$

Let us introduce

$$\Delta(x-y) = \Delta^+(x-y) + \Delta^-(x-y) \qquad (8.48)$$

Then equation (8.31) reads (at equal times)

$$[\varphi^o(x),\varphi^{o\dagger}(x)]_\pm = i\, d(\partial)\Delta(x-y)\ldots\text{type-two} \qquad (8.49)$$

$$[\varphi^o(x),\varphi^{o\dagger}(y)]_\pm = i\, d(\partial)\Delta^+(x-y)\ldots\text{type-one.} \qquad (8.50)$$

Equations (8.44) and (8.47) lead to

$$\rho = \begin{cases} +1 \ldots \text{boson} \\ -1 \ldots \text{fermion} \end{cases} \tag{8.51}$$

Hence the causality condition determines the statistics of particles obeying a type-two equation. Equations (8.49), (8.27) and (8.48) give

$$[\varphi^{o}(x,t), \frac{\partial \varphi^{o\dagger}(y,t)}{\partial t}]_{\pm} = i\,\delta(x-y) \tag{8.52}$$

and hence we have $\frac{\partial \varphi^{o\dagger}(x)}{\partial t} = \dot{\varphi}^{o\dagger}(x)$ as the canonical conjugate of $\varphi^{o}(x)$.

§9. The In and Out Fields

The asymptotic limit of a Heisenberg field is free. We understand this with two realizations; initially prepared states ($t \to -\infty$) can be chosen to be eigenstates of the free Hamiltonian (and hence are at least instanteously free); final states ($t \to +\infty$) although they may be thought to be spacially separated are still interacting for their wave packets have spread to a very large extent. However the density of the wave packets is ever decreasing as the expansion goes on and the net overlap between these final states, as time goes to infinity, is zero. They are then asymptotically free. This is the observation of R. Haag. We now wish to introduce such initial states (in-states) and final states (out-states).

We have from equation (8.21)

$$a(k) = \int d^3x\ u(k)^{\dagger} \exp(-ik\cdot\vec{x} + i\omega(k)t) \overleftrightarrow{\partial}_t \varphi^{o}(x) \qquad (9.1)$$

From equation (7.2) we introduce

$$a(k,t) = \int d^3x\ u(k)^{\dagger} \exp(-ik\cdot\vec{x} + i\omega(k)t) \overleftrightarrow{\partial}_t Z^{-1/2}[\psi(x) - \chi], \qquad (9.2)$$

and construct (analogous to equation (4.10))

$$\psi_g(t) \equiv \int \frac{d^3k}{(2\pi)^{3/2}}\, g(k)a(k,t) \qquad (9.3)$$

where $g(k)$ is a square-integrable function. Since wave packet states have well defined limits we can define $a^{in}(k)$ and $a^{out}(k)$ through

$$\lim_{t\to+\infty} \langle a|\psi_g(t)|b\rangle \equiv \langle a| \int d^3k\ g(k)a^{out}(k)|b\rangle \qquad (9.4)$$

$$\lim_{t\to-\infty} \langle a|\psi_g(t)|b\rangle \equiv \langle a| \int d^3k\, g(k)a^{in}(k)|b\rangle. \quad (9.5)$$

The operator [illegible](x) - χ) appearing in equation (9.2) is called the interpolating field for $a^{in}(k)$ and $a^{out}(k)$, all of its oscillating components die away except the one in phase with the free asymptotic particle.

We have, up to this point, introduced operators

$$a(k), a^{in}(k), a(k),$$
$$a^{\dagger}(k), a^{out}(k), a^{\dagger}(k). \quad (9.6)$$

We now make a correspondence between the in-fields and the physical fields,

$$a^{in}(k) = a(k)$$
$$a^{in\dagger}(k) = a^{\dagger}(k). \quad (9.7)$$

This implies the expansion coefficients of the dynamical map, equation (7.2), are retarded functions, ie.

$$F(x-y, x-z) = \theta(t_x - t_y)\theta(t_x - t_z)\bar{F}(x-y, x-z). \quad (9.8)$$

We then have

$$a^{out}(k) = a^{in}(k) + \text{higher order products of } a^{in}(k) \text{ and } a^{out}(k). \quad (9.9)$$

§10. The Reduction and L.S.Z. Formulas

All of the discussion contained in this paper thus far is completely in the spirit of the postulative formulation of field theory (without thermal freedom) of Lehmann, Symanzik and Zimmerman (L.S.Z. formulation).[9] With six basic postulates[10] the L.S.Z. formulation develops a full quantum field theory. The heart of all six postulates is contained in the above with the exception of some intricate mathematical considerations. We now will discuss the S-matrix and then the two relations which make the L.S.Z. formulation very useful in practical calculations, namely the reduction formula and the L.S.Z. formula.

Recall that at equal times, with equation (9.6), we have

$$[\varphi^{in}(x,t),\varphi^{in}(y,t)] = 0 \; ; \; [\dot{\varphi}^{in}(x,t),\varphi^{in}(y,t)] = -i\delta(x-y) \tag{10.1}$$

$$[\varphi^{out}(x,t),\varphi^{out}(y,t)] = 0 \; ; \; [\dot{\varphi}^{out}(x,t),\varphi^{out}(y,t)] = -i\delta(x-y). \tag{10.2}$$

The physical Fock space is complete to the extent that it includes all observable states and hence it includes all $\varphi^{in}(x,t)$ and $\varphi^{out}(x,y)$, we hence are justified (see reference 10, for a more rigorous justification based on the fact that the 'in' and 'out' operators form an irreducible operator ring) in assuming a unitary operator 'S' in the Hilbert space such that

$$\varphi^{out}(x,t) = S^{-1}\varphi^{in}(x,t)S. \tag{10.3}$$

Equivalently with the normalized wave packet expansion given by equation (8.18)

$$a^{out}(k) = S^{-1}a^{in}(k)S \tag{10.4}$$

$$a^{out\dagger}(k) = S^{-1}a^{in\dagger}(k)S. \tag{10.5}$$

Using equation (10.4)

$$0 = S^{-1}a^{in}(k)S|0\rangle, \tag{10.6}$$

this implies

$$a^{in}(k)S|0\rangle = 0. \tag{10.6b}$$

However, since we know $a^{in}(k)|0\rangle = 0$ we have that

$$S|0\rangle = |0\rangle \tag{10.6c}$$

if we assume the vacuum is unique. Here we have set the phase of S such that the phase of the vacuum on each side of equation (10.6c) is the same. We then have

$$|0\rangle_{in} = |0\rangle_{out} \equiv |0\rangle$$

We then may write

$$\begin{aligned} |a\rangle_{in} &= a(k_1)^{in\dagger}\ldots a(k_n)^{in\dagger}|0\rangle \\ &= S\, a^{out}(k_1)^{\dagger}S^{-1}S\ldots a(k_n)^{out\dagger}S^{-1}|0\rangle \\ &= S\, a^{out}(k_1)^{\dagger}\ldots a(k_n)^{out\dagger}|0\rangle \\ &= S|a\rangle_{out}, \end{aligned} \tag{10.7}$$

which leads to

$${}_{out}\langle a|b\rangle_{in} = {}_{out}\langle a|S|b\rangle_{out}, \tag{10.8}$$

Similiarly

$${}_{out}\langle a|b\rangle_{in} = {}_{in}\langle a|S|b\rangle_{in}. \tag{10.9}$$

Equations (10.8) and (10.9) say that matrix elements of the S-matrix are computed as the overlap between the in-states

and out-states. If there are no bound states our in and out states are complete and

$$S = \sum |a\rangle_{in\ out}\langle a| \tag{10.10}$$

Towards the derivation of the reduction formulas[11], we continue to work for simplicity with one, Hermitian, scalar field. Consider an in-out matrix element of a time ordered product of interpolating Heisenberg field operators

$$J_{ab} = {}_{out}\langle a|T(\psi(x_1)\ldots\psi(x_n)|b\rangle_{in}. \tag{10.11}$$

If we take

$$|b\rangle_{in} = |c,\alpha(k)\rangle_{in} \equiv \alpha^{in}(k)^{\dagger}|c\rangle_{in}, \tag{10.12}$$

we have

$${}_{out}\langle a|T(\psi(x_1)\ldots\psi(x_n)\alpha^{\dagger}(k)_{in}|c\rangle_{in}. \tag{10.13}$$

Using the asymptotic limit defined in equation (9.5), we have

$$J_{a,b} \equiv {}_{out}\langle a|T(\psi(x_1)\ldots\psi(x_n))\alpha^{\dagger}(k) \quad |c\rangle_{in} \tag{10.14}$$

$$= \lim_{t\to-\infty} {}_{out}\langle a|T(\psi(x_1)\ldots\psi(X_n))\psi_g^{\dagger}(t)|c\rangle_{in}$$

$$= \lim_{t\to=\infty} - {}_{out}\langle a|T(\psi(x_1)\ldots\psi(x_n)) \int \frac{d^3k}{(2\pi)^{3/2}} d^3x\ g(k)$$

$$Z^{-1/2}[\psi(x) - \chi)^{\dagger} \overleftrightarrow{\partial}_t u(k,\vec{x},t)|c\rangle_{in} \tag{10.15}$$

where we used the conjugate of equation (9.2).

We can write

$$J_{a,b} = \lim_{t\to-\infty} \int \frac{d^3k}{(2\pi)^{3/2}} d^3x\ {}_{out}$$

$$\langle a|T(\psi(x_1)\ldots\psi(x_n)\psi'(x)\overleftrightarrow{\partial}_t g(k)u(k,\vec{x},t)|c\rangle_{in} \tag{10.16}$$

where $\psi'(x) = Z^{-1/2}(\psi(x) - \chi)$. (10.17)

With the time limit we may certainly take the interpolating field $\psi'(x)$ under the T-product. Suppressing the momentum integration (understanding though that we are using wave packets) we write equation (10.16) with a time integration

$$J_{a,b} = -\int d^3x\, dt\, \frac{\partial}{\partial t}\, {}_{out}\langle a|T(\psi(x_1)\ldots\psi(x_n)\psi'_g \overset{\leftrightarrow}{\partial}_t u(k,\vec{x},t))|c\rangle_{in}$$
$$+ \lim_{t\to+\infty} \int d^3x\, {}_{out}\langle a|T(\psi(x_1)\ldots\psi(x_n)\psi'_g(x))\overset{\leftrightarrow}{\partial}_t u(k,\vec{x},t)|c\rangle_{in}. \quad (10.18)$$

The time limit of the second term of equation (10.18) means that the interpolating field $\psi'_g(x)^\dagger$ may be brought through the T-product and hence describes a one particle unconnected interaction hence we write,

$$(J_{a,b})_{connected} = -i \int d^3k\, dt$$
$${}_{out}\langle a|T(\psi(x_1)\ldots\psi(x_n)\psi'(x)^\dagger)|c\rangle_{in}\, \lambda(-\overset{\leftarrow}{\partial}_t)u(k,\vec{x},t)_{connected}. \quad (10.19)$$

Where we have used the fact that

$$\frac{\partial}{\partial t}\int d^3x\, f(x)^\dagger \overset{\leftrightarrow}{\partial}_t g(x) = -i\int d^3x\, f(x)^\dagger(\lambda(\partial) - \lambda(-\overset{\leftarrow}{\partial}))g(x); \quad (10.20)$$

Elimination of a quanta from the bra or a hole from the bra or ket leads to the following additional reduction formulas,

$$\langle a|T(\psi(x_1)\ldots\psi(x_n))|\alpha(k),c\rangle_{connected}$$
$$= -i\rho \int d^3x\, dt\, v(k,\vec{x},t)^\dagger \lambda(\partial)$$
$$\langle a|T(\psi(x_1)\ldots\psi(x_n)\psi'(x))|c\rangle_{connected} \quad (10.21)$$

$$\langle \alpha,\beta(k)|T(\psi(x_1)\ldots\psi(x_n))|b\rangle_{\text{connected}}$$
$$= -ie\int d^3x\,dt$$
$$\langle a|T(\psi'(x)^\dagger\psi(x_1)\ldots\psi(x_n)|b\rangle\lambda(-\overleftarrow{\partial})v(k,\vec{x},t)_{\text{connected}} \quad (10.22)$$

$$\langle \alpha(k),a|T(\psi(x_1)\ldots\psi(x_n)|b\rangle_{\text{connected}}$$
$$= -i\int d^3x\,dt\,u(k,\vec{x},t)^\dagger\lambda(\partial)$$
$$\langle a|T(\psi(x_1)\ldots\psi(x_n)\chi'(x))|b\rangle_{\text{connected}} \quad (10.23)$$

where all bra vectors are out-states and all kets are in-states.

Repeated use of equations (10.20) through (10.23) in order to exhaust the out-bra and in-ket vectors, leaving only a vacuum expectation value, leads to the L.S.Z. formula

$$T(\psi(x_1)\ldots\psi(x_n)) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-i)^{n+m}}{(n+m)!}\int d^4x_1\ldots d^4x_n d^4y_1\ldots d^4y_m$$
$$:[\varphi^{o\dagger}(x_1)\lambda(\partial^1_x)]\ldots[\varphi^{o\dagger}(x_n)\lambda(\partial^n_x)]$$
$$\langle 0|T[\varphi(x_1)\ldots\varphi(x_n)\psi(x_1)\ldots\psi(x_n)\varphi^\dagger(y_1)\ldots\varphi^\dagger(y_m))|0\rangle_{\text{connected}}$$
$$[\lambda(-\overleftarrow{\partial}^1_y)\varphi^o(y_1)]\ldots[\lambda(-\overleftarrow{\partial}^m_y)\varphi^o(y_m)]:, \quad (10.24)$$

where ':' means normal product. Equation (10.24) may be writen as (with $F(x) = \psi(x_1)\ldots\psi(x_n)$

$$F(x) = \langle 0|T(F(x):\exp[-i\int d^4\xi\{J^{o\dagger}(\xi)\varphi(\xi) + \varphi^\dagger(\xi)J^o(\xi)\}:)|0\rangle \quad (10.25)$$

where the normal product is with respect to the free physical fields φ^o and $\varphi^{o\dagger}$ and

$$J^{o\dagger}(x) = \varphi^{o\dagger}(x)\lambda(\partial) \quad (10.26)$$

$$J^o(x) = \lambda(-\overleftarrow{\partial})\varphi^o(x). \quad (10.27)$$

With matrix elements taken between physical particle states and not out, (bra) and in, (ket) states respectively; we have with $F(x) = 1$

$$S = :\langle 0|T\{\exp(-i \int d^4\xi\{J^{o\dagger}(\xi)\varphi(\xi) + \varphi^{\dagger}(\xi)J^{o}(\xi)\}\})|0\rangle: . \tag{10.28}$$

§11. Two-point Green's Functions

Following reference (2) closely, which contains a precise discussion of two-point functions, let us identify the causal, advanced and retarded two-point Green's functions as

$$G_c^{AB}(x-y) = \langle 0|T(A(x),B(y))|0\rangle \tag{11.1}$$

$$= \theta(t_x-t_y)\langle 0|A(x)B(y)|0\rangle + \sigma\,\theta(t_y-t_x)\langle 0|B(y)A(x)|0\rangle \tag{11.2}$$

$$G_a^{AB}(x-y) = \theta(t_x-t_y)\langle 0|[A(x),B(y)]_\sigma|0\rangle \tag{11.3}$$

$$G_r^{AB}(x-y) = -\theta(t_y-t_x)\langle 0|[A(x),B_y]_\sigma|0\rangle \tag{11.4}$$

We define the fourier transform of $G_c^{AB}(k)$ as, with $k=(\vec{k},\omega(k))$

$$G_c^{AB}(x-y) = \frac{i}{(2\pi)^4}\int d^4k\, \exp(ik(x-y)G_c^{AB}(k) \tag{11.5}$$

The sum rules in equations (I.8.39) and (I.8.40) give, for type-two fields

$$\langle 0|\varphi^o(x)\varphi^{o\dagger}(y)|0\rangle = i\,d(\partial)\Delta^+(x-y) \tag{11.6}$$

$$\langle 0|\varphi^{o\dagger}(y)\varphi^o(x)|0\rangle = -i\,\rho\,d(\partial)\Delta^-(x-y). \tag{11.7}$$

Using (I.8.24) we have

$$G_c^o(x-y) = i(\theta(t_x-t_y)d(\partial)\Delta^+(x-y) - \theta(t_y-t_x)d(\partial)\Delta^-(x-y)$$

$$= id(\partial)\Delta_c(x-y) + i[\theta(t_xt_y),d(\partial)]\Delta(x-y) \tag{11.8}$$

where we used equation (I.8.48) and

$$\Delta_c(x-y) = \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x) \tag{11.9}$$

With equations (I.8.27), (I.8.28) and (I.8.25) we have

$$\left(-\frac{\partial^2}{\partial t^2} - \omega^2(\nabla)\right)\Delta_c(x) = \delta^4(x) \tag{11.10}$$

and hence if for type-two field $\varphi^{o}(x)$ such that

$$\lambda(\partial)\varphi^{o}(x) = 0$$

and $d(\partial)$ has time derivatives only up to first order, equation (11.8) gives

$$G_c^o(x-y) = i\, d(\partial)\Delta_c(x-y) \tag{11.12}$$

For type-one fields satisfying equation (11.11)

$$G_c^o(x-y) = i\, \theta(t_x-t_y)\Delta^{+}(x-y) \tag{11.13}$$

and with equations (8.42,b) and (8.42,c) we have

$$(i\frac{\partial}{\partial t} - \varepsilon(\nabla)\Delta_c(x) = \delta^4(x). \tag{11.14}$$

When

$$\Delta_c(x) = \theta(t)\Delta^{+}(x)$$

equation (11.13) yields

$$G_c^o(x-y) = i\, d(\partial)\Delta_c(x-y) + i[\theta(t_x-t_y),d(\partial)]\Delta^{+}(x-y) \tag{11.16}$$

which gives, where $d(\partial)$ has no time derivatives

$$G_c^o(x-y) = i\, d(\partial)\Delta_c(x-y) \tag{11.17}$$

This then ends our discussion of quantum field theory without thermal degrees of freedom; it is by no means complete, but for our purpose it is complete enough. Quantum field theory without thermal degrees of freedom has been incredibly successful at describing high energy particle physics even though the reality it describes is always at a finite temperature.

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Part II: Finite Temperature Field Theory

§1. Thermo Field Dynamics

Thermo Field Dynamics[1,2,3] (TFD) is a generalization of field theory without thermal freedom (Part I) such that the statistical mechanical ensemble average is replaced by a thermal vacuum expectation value and hence becomes a real time average, allowing all field theoretic devices and calculation methods. TFD introduces another Hilbert space, say tilde space '~', which is identical but σ-commuting (operators A,B are said to σ-commute if $[A,B]_\sigma = AB - \sigma BA$, where $\sigma = \pm 1$) with the original non-tilde Hilbert space of Section I, ' '. The direct product of these two spaces forms the complete space of TFD and a representation of this space can be shown to form a c^*-algebra (part IV section II). The doubling of the number of degrees of freedom, via the tilde field, facilitates the above mentioned thermal vacuum expectation. The tilde operation is completely defined by the first three of six axioms of TFD[2]. They are as follows:

Define two sets of operators $a = \{A\}$ and $\tilde{a} = \{\tilde{A}\}$. Then

I. $$[A(t),\tilde{B}(t)]_\sigma = 0 \text{ for all } A \in a \text{ and } \tilde{B} \in \tilde{a}. \tag{1.1}$$

II. The one-to-one and onto mapping; say tilde conjugation, between a and $\tilde{a}$ obeys for $A,B \in a$ and $\tilde{A},\tilde{B} \in \tilde{a}$ and $c_1,c_2 \in$ c-number

a) $$[AB]^{\sim} = \tilde{A}\tilde{B} \tag{1.2}$$

b) $[c_1A + c_2B]^{\sim} = c_1^{*}\tilde{A} + c_2^{*}\tilde{B}$ (1.3)

c) $(\tilde{A})^{\dagger} = (A^{\dagger})^{\sim}$ (1.4)

III. For $A \in a$;

$\tilde{\tilde{A}} = \varepsilon A$ where ε is a sign factor (1.5)

to be discussed.

This last axiom does not support a c^{*}-algebra but as will be discussed in part IV there exist an "adjusted tilde operation"[4] which makes $\varepsilon = 1$ for all $A \in a$.

§2. The Thermal Vacuum

The need for the doubling of the degrees of freedom will become clear as we derive the form of the temperature dependent vacuum $|0(\beta)>$, which is defined by:

$$<A> = Z_\sigma^{-1}(\beta)\mathrm{Tr}[e^{-\beta H}A] \equiv <0(\beta)|A|0(\beta)>, \qquad (2.1)$$

where $Z_\sigma^{-1}(\beta)$ is such that $<1> = 1$.

Following closely H. Umezawa's original line of enquiry[5] we write from above:

$$<0(\beta)|A|0(\beta)> = Z_\sigma^{-1}(\beta) \sum_n <n|A|n>e^{-\beta\omega_n} \qquad (2.2)$$

expanding $|0(\beta)>$ in terms of $|n>$ as

$$|0(\beta)> = \sum_n |n>a_n(\beta) \qquad (2.3)$$

we find

$$a_n(\beta)^* a_m(\beta) = Z_\sigma^{-1}(\beta)e^{-\beta\omega_n}\delta_{nm}. \qquad (2.4)$$

Sense can only be made of this if $a_n(\beta)$ is interpreted as a vector and not a c-number. Equation (2.4) then gives the orthogonality condition for elements of the vector space $\{a(\beta)\}$. Due to equation (2.3) this new space $\{a_n(\beta)\}$ is orthogonal to the space $\{|n>\}$ and together these two spaces span the space $\{|0(\beta)>\}$.

Define;

$$a_n(\beta) = |\tilde{n}>\, e^{-\beta\omega_n/2} Z_\sigma^{-1/2}(\beta) \qquad (2.5)$$

where $|\tilde{n}>$ is a vector in the tilde Hilbert space which can be characterized by:

$$\tilde{H}|\tilde{n}> = \omega_n|\tilde{n}> \qquad (2.6)$$

$$\langle\tilde{n}|\tilde{m}\rangle = \delta_{nm} \tag{2.7}$$

$$\tilde{a}^{\dagger}(k)\tilde{a}(k)|\tilde{n}\rangle = n(k)|\tilde{n}\rangle \tag{2.8}$$

Note by definition the eigen-energy appearing in equation (2.6) is the same as equation 5.1 of Part I. Substituting equation 2.5 into equation 2.3 we get;

$$|0(\beta)\rangle = Z_{\sigma}^{-1/2}(\beta)\sum_{n} |n,\tilde{n}\rangle e^{-\beta\omega_n/2} \tag{2.9}$$

Where we have denoted the space $|n\rangle \times |\tilde{m}\rangle = |n,\tilde{m}\rangle$.

For boson like operator A:

$$\langle\tilde{m},n|A|n',\tilde{m}'\rangle = \langle n|A|n'\rangle\delta_{mm'} \tag{2.10}$$

$$\langle\tilde{m},n|\tilde{A}|n',\tilde{m}'\rangle = \langle\tilde{m}|\tilde{A}|\tilde{m}'\rangle\delta_{nn'} \tag{2.11}$$

ie. A and $\tilde{A}$ only act on vectors of their respective Hilbert spaces.

We make the definition

$$\langle m|F|n\rangle \equiv \langle\tilde{n}|\tilde{F}^{\dagger}|\tilde{m}\rangle \tag{2.12}$$

With equation (2.9) we may write the thermal vacuum expectation of an non-tilde operator as;

$$\langle 0(\beta)|A|0(\beta)\rangle = Z_{\sigma}^{-1}(\beta)\sum_{n}\sum_{m} e^{-\beta\omega_n/2}\langle\tilde{n},n|A|m,\tilde{m}\rangle \tag{2.13}$$

$$= Z_{\sigma}^{-1}(\beta)\sum_{n}\sum_{m} e^{-\beta\omega_n/2}e^{-\beta\omega_m/2}\langle n|A|m\rangle\langle\tilde{n}|\tilde{m}\rangle \tag{2.14}$$

using equation (2.7) we have

$$\langle 0(\beta)|A|0(\beta)\rangle = Z_{\sigma}^{-1}(\beta)\sum_{n} e^{-\beta\omega_n}\langle n|A|n\rangle \tag{2.15}$$

which is equation (2.1).

In the simplest sense we have introduced a tilde space so that only the diagonal elements of A are picked up in <A>. More fundamentally, as will be shown, the physical mapping between a tilde and non-tilde operator (the thermal state condition) demands a new parameter enter the theory, this parameter is temperature. The initial condition for this mapping allows one to include the initial thermal condition in the formalism. This initial condition can be static or dynamic developing into an equilibrium or non-equilibrium formalism respectively. This will be made more clear in Part III. The uniqueness and strength of TFD as a finite temperature quantum field theory is that it is based upon thermal states built upon the thermal vacuum. These excited states form a linear vector space and are responsible for the fact that TFD accommodates a thermal Wick's formula and a Feynman-type diagram method.

§3. Normalization of the Thermal Vacuum

Let us consider a system with Hamiltonian $H(a^{\dagger}_{(k)}a_{(k)})$ where

$$[a(k),a^{\dagger}(k')]_{\sigma} = \delta(k-k') \tag{3.1}$$

$$[a(k),a(k')]_{\sigma} = [a^{\dagger}(k),a^{\dagger}(k')]_{\sigma} = 0 \tag{3.2}$$

with vacuum state $|0\rangle$ such that

$$H(a(k)^{\dagger}a(k))|0\rangle = 0 \tag{3.3}$$

Any excited eigen state of this system can be expressed as a vector in the Fock space generated by the cyclic action of operators $a^{\dagger}(k)$ on the vacuum $|0\rangle$. Hence the Fock space is spanned by the vectors $a^{\dagger}(k)^{n}|0\rangle$.

$$H(a^{\dagger}(k))^{n}|0\rangle = n\,\omega(k)\,(a^{\dagger}(k))^{n}|0\rangle.$$

We now introduce the tilde system's algebra in a similar way. The tilde system has a Hamiltonian $\tilde{H}(\tilde{a}^{\dagger}(k)\tilde{a}(k))$ where

$$[\tilde{a}(k),\tilde{a}(k')^{\dagger}]_{\sigma} = \delta(k-k') \tag{3.4}$$

$$[\tilde{a}(k,\tilde{a}(k')]_{\sigma} = [\tilde{a}^{\dagger}(k),\tilde{a}^{\dagger}(k')]_{\sigma} = 0 \tag{3.5}$$

with vacuum state $|\tilde{0}\rangle$ such that

$$\tilde{H}(\tilde{a}^{\dagger}(k)\tilde{a}(k))|\tilde{0}\rangle = 0 \tag{3.6}$$

Excited tilde states form vectors in the tilde Fock space. These vectors are generated by cyclic action of $\tilde{a}^{\dagger}(k)$ on the vacuum $|\tilde{0}\rangle$. Hence the tilde Fock space is spanned by $(\tilde{a}^{\dagger}(k))^{n}|\tilde{0}\rangle$ and

$$\tilde{H}(\tilde{a}^{\dagger}(k))^{n}|\tilde{0}\rangle = n\,\omega_{k}(\tilde{a}^{\dagger}(k))^{n}|\tilde{0}\rangle. \qquad (3.7)$$

We may then write;

$$|0(\beta)\rangle = z_{\sigma}^{-1/2}(\beta)\, e^{-\beta H/2} \exp(\sum_{k} a^{\dagger}(k)\tilde{a}^{\dagger}(k))|0\tilde{0}\rangle. \qquad (3.8)$$

The normalization $z_{\sigma}^{-1/2}(\beta)$ can be obtained by recalling that the number density 'n(k)' is such that;

$$\begin{aligned} n(k) &= \frac{1}{e^{\beta\omega'} + \sigma} \\ &= \langle 0(\beta)|a(k^{\dagger})a(k)|0(\beta)\rangle \\ &= z_{\sigma}^{-1}(\beta)\langle 0\tilde{0}|\exp(\sum_{k} \tilde{a}(k)a(k))e^{-\beta H/2}\, a^{\dagger}(k)a(k)e^{-\beta H/2} \\ &\exp(\sum_{k} a(k)^{\dagger}\tilde{a}(k)^{\dagger}|0\tilde{0}\rangle \\ &= z_{\sigma}^{-1}(\beta)e^{-\beta\omega'} \end{aligned} \qquad (3.10)$$

where $\omega' = n\omega(k,\beta)$. Then we have, dropping the prime;

$$z_{\sigma}^{-1}(\beta) = \frac{1}{1 + \sigma e^{-\beta\omega}}\,. \qquad (3.11)$$

If we denote

$$u(\beta) = (1 + \sigma\, e^{-\beta\omega})^{-1/2} \qquad (3.12)$$

$$v(\beta) = e^{-\beta\omega/2}(1 + \sigma\, e^{-\beta\omega})^{-1/2} \qquad (3.13)$$

we see that

$$u(\beta)^{2} + v(\beta)^{2} = 1 \qquad \text{for } \sigma = 1, \text{ fermion} \qquad (3.14)$$

$$u(\beta)^{2} - v(\beta)^{2} = 1 \qquad \text{for } \sigma = -1, \text{ boson} \qquad (3.15)$$

Because of equations (3.14) and (3.15) we will denote

$$u(\beta)_{\text{fermion}} = \cos\theta(\beta)$$

$$u(\beta)_{\text{boson}} = \cosh\theta(\beta)$$

$$v(\beta)_{\text{fermion}} = \sin\theta(\beta)$$

$$v(\beta)_{boson} = \sinh \theta(\beta).$$

We consider bose oscillators first of all ($\sigma = -1$), we may write

$$|0(\beta)\rangle = (1 - e^{-\beta\omega})\exp(e^{-\beta\omega/2} a^\dagger \tilde{a}^\dagger)|0\tilde{0}\rangle \quad (3.16)$$

$$= \exp(\theta(\beta)(a^\dagger \tilde{a}^\dagger - \sigma' a\tilde{a}))|0\tilde{0}\rangle \quad (3.17)$$

$$\equiv \exp(-i\, G_B)|0\tilde{0}\rangle \quad (3.18)$$

where $\theta(\beta)$ is such that:

$$\cosh \theta(\beta) = (1 - e^{-\beta\omega})^{-1/2}, \quad \sinh \theta(\beta) = e^{-\beta\omega/2}(1 - e^{-\beta\omega})^{-1/2} \quad (3.19)$$

and σ' may be chosen as + or - unity. Equation (3.17) shows $\exp(-iG_B)$ is unitary and hence suggests a Bogoliubov transformation

$$a(\beta) \equiv \exp(-iG_B)a \exp(iG_B)$$

$$= a \cosh \theta(\beta) - \sigma' \tilde{a}^\dagger \sinh \theta(\beta) \quad (3.20)$$

$$\tilde{a}(\beta) \equiv \exp(-iG_B)\tilde{a} \exp(iG_B)$$

$$= \tilde{a} \cosh \theta(\beta) - \sigma' a^\dagger \sinh \theta(\beta) \quad (3.21)$$

which invert to give

$$a = a(\beta) \cosh \theta(\beta) + \sigma' \tilde{a}^\dagger(\beta) \sinh \theta(\beta) \quad (3.22)$$

$$\tilde{a} = \tilde{a}(\beta) \cosh \theta(\beta) + \sigma' a^\dagger(\beta) \sinh \theta(\beta) \quad (3.23)$$

By virtue of the fact that the transformation of our annihilation operators into thermal annihilation operators is a Bogoliubov transformation, our thermal operators obey the same commutation relations as did our non-thermal ones,

$$[a(\beta), a^\dagger(\beta)]_- = [\tilde{a}(\beta), \tilde{a}^\dagger(\beta)]_- = 1 \quad (3.24)$$

all others are zero.

We see from equation (3.18) and (3.20) that

$$a(\beta)|0(\beta)\rangle = \tilde{a}(\beta)|0(\beta)\rangle = 0 \tag{3.25}$$

We now see $|0(\beta)\rangle$'s vacuum nature. Using equation 3.20 and 3.21 we have

$$\begin{aligned} a|0(\beta)\rangle &= \sigma' \tanh \theta(\beta)\tilde{a}^\dagger|0(\beta)\rangle \\ &= \sigma' e^{-\beta\omega/2}\tilde{a}^\dagger|0(\beta)\rangle \end{aligned} \tag{3.26}$$

$$a^\dagger|0(\beta)\rangle = \sigma' e^{\beta\omega/2}\,\tilde{a}|0(\beta)\rangle. \tag{3.27}$$

Now for fermi oscillators ($\sigma = +1$) we have

$$\begin{aligned} |0(\beta)\rangle &= (1 + e^{-\beta\omega})^{-1/2}(1 + e^{-\beta\varepsilon/2}b^\dagger\tilde{b}^\dagger)|0\tilde{0}\rangle \\ &= \exp(\theta(\beta)(b^\dagger\tilde{b}^\dagger - \sigma'\tilde{b}b))|0\tilde{0}\rangle \qquad (3.28) \\ &\equiv \exp(-iG_F)|0\tilde{0}\rangle \qquad (3.29) \end{aligned}$$

where $\theta(\beta)$ is such that:

$$\cos\theta(\beta) = (1 + e^{-\beta\omega})^{-1/2},\ \sin\theta(\beta) = e^{-\beta\omega/2}(1 + e^{-\beta\omega})^{1/2} \tag{3.30}$$

and as with the boson case we define

$$\begin{aligned} b(\beta) &\equiv \exp(-iG_F)b\exp(iG_F) \\ &= b\cos\theta(\beta) - \sigma'\tilde{b}^\dagger\sin\theta(\beta) \end{aligned} \tag{3.31}$$

$$\begin{aligned} \tilde{b}(\beta) &\equiv \exp(-iG_F)\tilde{b}\exp(iG_F) \\ &= \tilde{b}\cos\theta(\beta) + \sigma' b^\dagger\sin\theta(\beta). \end{aligned} \tag{3.32}$$

Which invert to give

$$b(k) = b(k,\beta)\cos\theta(\beta) + \sigma'\tilde{b}(k,\beta)^\dagger\sin\theta(\beta) \tag{3.33}$$

$$\tilde{b}(k) = \tilde{b}(k,\beta)\cos\theta(\beta) - \sigma' b(k,\beta)^\dagger\sin\theta(\beta). \tag{3.34}$$

It follows that

$$[b(\beta),b^\dagger(\beta)]_+ = [\tilde{b}(\beta),\tilde{b}^\dagger(\beta)]_+ = 1 \tag{3.35}$$

with all others zero and that

$$b(\beta)|0(\beta)\rangle = \tilde{b}(\beta)|0(\beta)\rangle = 0. \tag{3.36}$$

We have, using 3.31 and 3.32 that:

$$b|0(\beta)\rangle = +\sigma' \tan\theta(\beta)\tilde{b}^{\dagger}|0(\beta)\rangle \tag{3.37}$$

$$= +\sigma' e^{-\beta\omega/2}\tilde{b}^{\dagger}|0(\beta)\rangle \tag{3.38}$$

$$b^{\dagger}|0(\beta)\rangle = -\sigma' e^{\beta\omega/2}\tilde{b}|0(\beta)\rangle$$

Relations (3.26), (3.27), (3.37) and (3.38) show that the quanta eliminated by a non-tilde annihilation operator acting on the thermal vacuum is equivalent to the quanta created by a tilde creation operator. We can therefore consider the $\tilde{a}(k)$ quantum to be a hole of the a(k) quanta.

§4. The Thermal State Condition

Using equations (3.26), (3.27), (3.35) and (3.36) we see that for arbitrary physical fields φ consisting of $a(k)$, $a^\dagger(k)$, $\tilde{a}(k)$, $\tilde{a}^\dagger(k)$

$$\varphi|0(\beta)> = \varepsilon_F \exp(\beta/2(H - \tilde{H}))\tilde{\varphi}^\dagger|0(\beta)> \qquad (4.1)$$

where ε_F is a sign function depending on the nature of φ (ie. it depends on σ' of the constituents of φ and on the sign of equation 1.5). In part IV we will see this factor can be eliminated by use of an "adjusted" tilde operation.

In order to see how ε_F is determined let us consider $\varphi = a^\dagger(k_1)\ldots a^\dagger(k_n)a(q_1)\ldots a(q_m)$ where $a(k)$ is a fermion operator. We wish to express φ as $\tilde{\varphi}^\dagger$ so we first reverse the order of the creation and annihilation operators. This implies η_F exchanges where η_F is given by;

$$\eta_F = \frac{1}{2}(n + m)(n + m - 1) \qquad (4.2)$$

Amidst our creation operators let us say that there are ν with $\sigma' = -1$ and $(n - \nu)$ with $\sigma' = 1$, and amidst the annihilation operators let us say there are μ with $\sigma' = -1$ and $(m - \mu)$ with $\sigma' = 1$. From equations (3.35) and (3.36) we see that creation operators with $\sigma' = 1$ and annihilation operators with $\sigma' = -1$ add an extra minus sign, hence

$$\varepsilon_F = (-1)^{\eta_F}(-1)^{\eta-\nu}(-1)^{\mu} = (-1)^{\eta_F - n}(-1)^{\mu-\nu} \qquad (4.3)$$

which may be written:

$$\varepsilon_F = (-1)^{1/2\left((n-\nu)-(m-\mu)+(m-\nu)\right)\left((n-\nu)-(m-\mu)+(\mu-\nu)+1\right)} \qquad (4.4)$$

or by equation 4.2

$$\varepsilon_F = (-1)^{1/2\ F(F+1)} \tag{4.5}$$

where the fermion number operator F is given by

$$F = \int d^3k\ \sigma'[a^\dagger(k)a(k) - \tilde{a}^\dagger(k)\tilde{a}(k)] \tag{4.6}$$

The Hermitian conjugate of equation (4.1) yields

$$\langle 0(\beta)|\varphi = \langle 0(\beta)|\tilde{\varphi}^\dagger \exp(\frac{\beta}{2}\ (H - \tilde{H}))\varepsilon_F \tag{4.7}$$

This relation can be further generalized for Heisenberg operators consisting of φ. These relations (4.1) and (4.7) are known as the equilibrium thermal state condition. Recall our previous discussion about such a condition, here in TFD we have an equilibrium formalism and we see that the thermal state condition is a stationary one. It is this correlation between tilde and non-tilde fields which is characteristic of an equilibrium situation at temperature β. To change the form of the thermal state condition is to change the correlations and nature of the system under study. We will return to this point in part III.

§5. Conclusions and the Thermal Vacuum as a Pure Quantum Mode

We have thus far in this part of this paper presented the form of $|0(\beta)>$, the thermal vacuum, determined its normalization and have shown that thermal vacuum expectation values do indeed equal statistical mechanical ensemble averages. We have found the Bogoliubov transformation which gives us the thermal annihilation and creation operators which act on the thermal vacuum in terms of tilde and non-tilde annihilation and creation operators. Lastly we have derived the relation between tilde and non-tilde operators which is the thermal state condition.

We will now show $|0(\beta)>$ to be a pure quantum state, a remarkable fact. We will then have the basis for a complete field theoretic formalism for equilibrium finite temperature systems.

A discussion about the concept of "state" in statistical mechanics, although it may seem lengthy, will shed light on the very different concept of the thermal state in TFD. Let us consider an isolated system in which we consider only those of its "situations" (it is too early in this discussion to use the word state) which have a definite energy value ε. Such a situation can be defined by a normalized eigen-function 'ψ_i' of the system's Hamiltonian ($H\,\psi_i = \varepsilon\,\psi_i$). The set of all such $\{\psi_i\}$ for all ε values

represents a basis of a complex Hilbert space. Elements of this space specify the situation of the system with maximum completeness (hence the index i). Consider the sub-set of the Hilbert space whose elements have a common eigen energy ε, $\{\psi_\varepsilon^j$; say $j = 1...s\}$. Any situation of the system with energy ε can be represented by

$$\psi_\varepsilon = \sum_{j=1}^{s} \gamma_j \psi_\varepsilon^j \qquad (5.1)$$

where $\gamma_j \in C$ and $\sum_{j=1}^{s} |\gamma_j|^2 = 1$

ie. $\{\psi_\varepsilon^j\}$ is isomorphic to the complex sphere. If only the value of energy 'ε' of a system is known then the phase of the function 'ψ_ε' describing this situation of state is undetermined, as is any phase function ($f(\psi_\varepsilon)$) which is the eigen function of an observable of the system in energy state ε. We must be content to draw information from the real sphere which is derived by assuming that the phases (ω) are uniformly distributed in $[0, 2\pi)$, independent of their moduli. We call this information the expectation of our phase function $f(\psi_\varepsilon)$,

$$\langle f(\psi_\varepsilon)\rangle = \int_{\{\psi_\varepsilon\}} f(\psi_\varepsilon) d\omega \qquad (5.2)$$

$$= \int_{\{\psi_\varepsilon\}} f\left(\sum_{k=1}^{s} \gamma_k \psi_\varepsilon^k\right) d\omega. \qquad (5.3)$$

If F is the operator whose eigenfunctions are observables 'f', we then have

$$\langle F(\psi_\varepsilon)\rangle = \int_{\{\psi_\varepsilon^j\}} (\psi_\varepsilon, F\,\psi_\varepsilon)d\omega = \frac{1}{s}\sum_{j=1}^{s} (\psi_s^j, F\,\psi_\varepsilon^j) \quad (5.4)$$

For a system with a boltzmann distribution we then have

$$\langle F\rangle = \sum_i \frac{e^{-\beta\varepsilon_i}\langle f(\psi_{\varepsilon_i})\rangle}{\sum_i e^{-\beta\varepsilon_i}} \quad (5.5)$$

where the expansion coefficients $\langle f(\psi_{\varepsilon_i})\rangle$ are real numbers. In statistical mechanics the "state of a system" is just this functional over all observables of the system. Clearly it is a mixed state. When we speak of an experimentally prepared state in statistical mechanics we are speaking of the preparation of a system so as to return a given set of expectation values.

In TFD we could except a similar definition for a state, that being the functional

$$\langle F\rangle = \langle 0(\beta)|F|0(\beta)\rangle \quad (5.6)$$

over all observables F. This would be a mixed state and allow no field theoretical consideration, but if we write

$$\begin{aligned}\langle F\rangle &= \langle 0(\beta)|F|0(\beta)\rangle \\ &= Z_\sigma^{-1}(\beta)\langle\tilde{00}|\exp\Big(\sum_k \tilde{a}(k')a(k')\Big)e^{-\beta H/2}F\,e^{-\beta H/2}\cdot \\ &\qquad \exp\Big(\sum_k a^\dagger(k)\tilde{a}^\dagger(k)\Big)|\tilde{00}\rangle \quad (5.7)\\ &= Z_\sigma^{-1}(\beta)\sum_i e^{-\beta\varepsilon_i}\langle 0\tilde{0}|\Big(\sum_k \tilde{a}(k')a(k')\Big)^i F\Big(\sum_k a(k)^\dagger\tilde{a}(k)^\dagger\Big)^i|0\tilde{0}\rangle\end{aligned}$$

$$= Z_\sigma^{-1}(\beta) \sum_i e^{-\beta\epsilon_i} \langle 0|(a(k))^i F(a(k^\dagger))^i|0\rangle \qquad (5.8)$$

we see that the expectation of F is taken between pure quantum states $a(k)^\dagger|0\rangle$. Any phases carried by the thermal vacuum have no effect on any observable quantity, as can be seen by equation (5.7). The phase of each $|n_i n_i'\rangle$ element in the thermal vacuum is amalgamated with the creation operator which creates such a state from $|\tilde{0}0\rangle$. All of our previous definitions of annihilation and creation operators take the phase of each term in the expansion of $|0(\beta)\rangle$ to be the same and equal to zero. In the already mentioned adjusted annihilation and creation operators the phase is taken to be $-\pi/2$ for all fermionic excitations and zero for all bosonic excitations. In other words since the thermal vacuum is a linear combination of orthonormal base vectors it is unique, once the phases are chosen. The expansion given in equation (2.9) gives the desired expectation value of statistical mehanics. Once the physical annihilation and creation operators are given a representation this vacuum state is completely defined, and hence considered a pure quantum state. Technically any unusual phase distribution of the components of $|0(\beta)\rangle$ can be handled by an equivalently unusual definition of the annihilation and creation operators. Clearly from its form $|0(\beta)\rangle$ is properly symmetrized but has no correlations. The correlations come

into the theory through the thermal state condition which relates the tilde and non-tilde fields. It is worthwhile to note that any type of correlation can be brought in through the thermal state condition although they all go under the heading "thermal". It follows that any completely defined excitation of the thermal vacuum is also a pure quantum state and hence we are justified in building a field theory upon these thermal states.

This way of thinking about the state of a thermal system and use of a σ-communtant field to accomplish a definition for such a state, lies at the heart of the tremendous contribution to quantum field theory which is thermo-field dynamics.

§6. Generalization of the Momentum Representation and Two Point Green's Function to Finite Temperature

Using the axioms introduced in the first two sections of this part we can generalize the momentum representation and two point Green's functions. Consider for example a free, type-one field, $\varphi^{o}(x)$

$$\lambda(\partial)\varphi^{o}(x) = (i\frac{\partial}{\partial t} - \omega(\nabla))\varphi^{o}(x) = 0. \tag{6.1}$$

We then have

$$= \int d^{4}x\ \varphi^{o}(x)^{\dagger}(i\frac{\partial}{\partial t} - \omega(\nabla))\varphi^{o}(x). \tag{6.2}$$

Now applying our tilde conjugation rules we have, with $\omega^{*}(\nabla) = \omega(-\nabla)$,

$$= \int d^{4}x\ \tilde{\varphi}^{o}(x)^{\dagger}(-i\frac{\partial}{\partial t} - \lambda(-\nabla))\tilde{\varphi}^{o}. \tag{6.3}$$

The Hamiltonian

$$H_{o} = \int d^{3}x\ \varphi^{o}(x)^{\dagger}\omega(\nabla)\varphi^{o}(x) \tag{6.4}$$

has tilde conjugate

$$\tilde{H}_{o} = \int d^{3}x\ \tilde{\varphi}^{o}(x)^{\dagger}\omega(-i\nabla)\tilde{\varphi}^{o}(x) \tag{6.5}$$

The free field operator

$$\varphi^{o}(x) = (2\pi)^{-3/2}\int d^{3}k\ a(k)\exp(ik\cdot\vec{x} - i\omega(k)t) \tag{6.6}$$

has tilde conjugate

$$\tilde{\varphi}^{o}(x) = (2\pi)^{-3/2}\int d^{3}k\ \tilde{a}(k)\exp(-ik\cdot\vec{x} + i\omega(k)t). \tag{6.7}$$

Putting equations (6.6) and (6.7) into equations (6.4) and (6.5), we obtain

$$H_{o} = \int d^{3}k\ \omega(k)a(k)^{\dagger}a(k) \tag{6.8}$$

$$\tilde{H}_{o} = \int d^{3}k\ \omega(k)\tilde{a}(k)^{\dagger}\tilde{a}(k) \tag{6.9}$$

In the above, $\alpha(k)$ and $\tilde{\alpha}(k)$ are related to $\alpha(k,\beta)$ and $\tilde{\alpha}(k,\beta)$ through equations (3.22) and (3.23) for bosons and equations (3.33) and (3.34) for fermions. For example for a fermionic field, associating the positive frequency part of equation (6.6) with an annihilation operator $a(k)$ and the negative frequency part of equation (6.6) with a creation operator $b(k)^\dagger$, we have

$$\varphi^o(x) = (2\pi)^{-3/2}\int d^3k \left(a(k)\theta(\omega(k)) + b^\dagger(k)\,\theta(-\omega(k))\right) \exp(-i\omega(k) + ik\cdot\vec{x}) \tag{6.10}$$

Now with equation (3.33) and (3.34) we have (letting $\sigma' = +1$)

$$a(k) = \cos\theta(\beta)a(k,\beta) + \sin\theta(\beta)\tilde{a}^\dagger(k,\beta) \tag{6.11}$$

$$b^\dagger(k) = \cos\theta(\beta)\tilde{b}(k,\beta) - \sin\theta(\beta)b^\dagger(k,\beta) \tag{6.12}$$

Hence we write

$$\begin{aligned} &a(k)\theta(\omega(k)) + b^\dagger(k)\theta(-\omega(k)) \\ &= \cos\theta(\beta)\left(a(k,\beta)\theta(\omega(k)) - \tilde{b}(k,\beta)\theta(-\omega(k))\right) \\ &+ \sin\theta(\beta)\left(\tilde{a}^\dagger(k,\beta)\theta(\omega(k)) + b^\dagger(k,\beta)\theta(-\omega(k))\right) \qquad (6.13) \\ &\equiv \cos\theta(\beta)\alpha(k,\beta) + \sin\theta(\beta)\tilde{\alpha}^\dagger(k,\beta) \qquad (6.14) \end{aligned}$$

The last identification of $\alpha(k,\beta)$ and $\tilde{\alpha}^\dagger(k,\beta)$ makes equations (6.10) and (6.6) equivalent and $\alpha(k,\beta)$ $(\alpha^\dagger(k,\beta))$ is our quasi-particle annihilation (creation) operator.

We now introduce for simplicity the thermal doublet notation

$$(F^\alpha;\ \alpha = 1,2) = \begin{matrix} F^2 \\ F^2 \end{matrix} = \begin{matrix} P_1(F) \\ P_2(\tilde{F}^\dagger) \end{matrix} \tag{6.15}$$

When F = aAB...C, (where a is a c-number and AB...C are operators) we have

$$F^{\alpha} = a\, P_{\alpha}(A^{\alpha}B^{\alpha}\ldots C^{\alpha}) \tag{6.15b}$$

where

$$P_{\alpha}(A^{\alpha}B^{\alpha}\ldots C^{\alpha}) = \begin{cases} A^{1}B^{1}\ldots C^{1} & \text{for } \alpha = 1 \\ C^{2}\ldots B^{2}A^{2} & \text{for } \alpha = 2\,. \end{cases} \tag{6.15c}$$

Note that for a time ordered product F = T(AB...C) that $P_1(A'B'\ldots C')$ returns the time ordered product of the first elements of the thermal doublet but $P_2(A^2B^2\ldots C^2)$ returns the anti-time ordered product of the second elements of the thermal doublet.

The Green's functions consideration of part I, section 11 can be easily generalized for finite temperature situations. Let us take for example a free boson field $\varphi^{o}(x)$ of type one. The causal Green's function is given by equation (I.11.2) with its tilde conjugate as

$$G_c^{\alpha\beta}(x-y) = \langle 0(\beta)|T(\varphi^{\alpha}(x),\varphi^{\beta}(y)^{\dagger})|0(\beta)\rangle \tag{6.16}$$

$$= \theta(t_x-t_y)\langle 0(\beta)|\varphi^{\alpha}(x)\varphi^{\beta}(y)^{\dagger}|0(\beta)\rangle + \theta(t_y-t_x)\langle 0(\beta)|\varphi^{\beta}(y)^{\dagger}\varphi^{\alpha}(x)|0(\beta)\rangle \tag{6.17}$$

Note that these are (2 × 2) matrix equations. We can write the fourier form as

$$G_c^{\alpha\beta}(x-y) = \frac{i}{(2\pi)^4}\int d^4k\, \exp\left[(ik\cdot\vec{x}-\vec{y}) - i\,k_o(t_x-t_y)\right]G_c^{\alpha\beta}(k) \tag{6.18}$$

Now with Bugoliubov transformations (3.22) and (3.33) we have for equations (6.6) and (6.7)

$$\varphi^{o}(x) = (2\pi)^{-3/2} \int d^3k\ \alpha(k,\beta)\cosh\theta(\beta) + \tilde{\alpha}^{\dagger}(k,\beta)\sinh\theta(\beta) \tag{6.19}$$

$$\tilde{\varphi}^{o}(x) = (2\pi)^{-3/2} \int d^3k\ \tilde{\alpha}(k,\beta)\cosh\theta(\beta) + \alpha^{+}(k,\beta)\sinh\theta(\beta) \tag{6.20}$$

Using

$$\theta(t_x - t_y) = \lim_{\delta \to +0} \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega\ \frac{1}{\omega - i\delta} \exp(-i\omega(t_x - t_y)), \tag{6.21}$$

we have with equations (6.19) and (6.20) substituted into equation (6.17) the identification, by equation (6.18), that

$$G_c^{ab}(k) =$$

$$\begin{pmatrix} \dfrac{\cosh^2\theta(\beta)}{k_o - \omega + i\delta} - \dfrac{\sinh^2\theta(\beta)}{k_o - \omega - i\delta}, & \dfrac{\cosh\theta(\beta)\sinh\theta(\beta)}{k_o - \omega + i\delta} - \dfrac{\cosh\theta(\beta)\sinh\theta(\beta)}{k_o - \omega - i\delta} \\ \dfrac{\cosh\theta(\beta)\sinh\theta(\beta)}{k_o - \omega + i\delta} - \dfrac{\cosh\theta(\beta)\sinh\theta(\beta)}{k_o - \omega - i\delta}, & \dfrac{\sinh^2\theta(\beta)}{k_o - \omega + i\delta} - \dfrac{\cosh^2\theta(\beta)}{k_o - \omega - i\delta} \end{pmatrix} \tag{6.22}$$

which can be written

$$G_c^{\alpha\beta}(k) = \begin{pmatrix} \cosh\theta(\beta), & \sinh\theta(\beta) \\ \sinh\theta(\beta), & \cosh\theta(\beta) \end{pmatrix} \begin{pmatrix} (k_o - \omega + i\delta)^{-1}, & 0 \\ 0, & (k_o - \omega\ i\delta)^{-1} \end{pmatrix} \times$$

$$\begin{pmatrix} \cosh\theta(\beta), & \sinh\theta(\beta) \\ \sinh\theta(\beta), & \cosh\theta(\beta) \end{pmatrix} \tag{6.23}$$

$$= u_{\overline{\text{Boson}}}(\theta(\beta)\tau(k_o - \omega + i\delta\tau)^{-1} u_{\text{Boson}}(\theta(\beta)) \tag{6.24}$$

where $$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.25}$$

and

$$u_{Boson}(\theta(\beta)) = \begin{pmatrix} \cosh\theta(\beta), & \sinh\theta(\beta) \\ \sinh\theta(\beta), & \cosh\theta(\beta) \end{pmatrix}. \tag{6.26}$$

This is a most incredible result; for with equations (I.11.15) and (I.11.17) with (I.8.42) we see that this thermal causal Green's function is just the zero temperature one sandwiched between a "thermal distribution matrix" with the addition of the τ matrix to account for anti-time ordering. We have for the advanced and retarded two point functions of equations (I.11.3) and (I.11.4)

$$G_r^{\alpha\beta}(k) = \frac{\tau^{\alpha b}}{k_o - \omega + i\delta} \tag{6.27}$$

$$G_a^{\alpha\beta}(k) = \frac{\tau^{\alpha\beta}}{k_o - \omega + i\delta}. \tag{6.28}$$

A similar result follows for fermionic type-one fields with

$$u_{Fermion}(\theta(\beta)) = \begin{pmatrix} \cos\theta(\beta) & \sin\theta(\beta) \\ -\sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix} \tag{6.29}$$

$$G_c^{\alpha\beta}(k) = u_F(\theta(\beta))(k_o - \omega + i\delta\tau)^{-1}u_F^{-1}(\theta(\beta)); \tag{6.30}$$

and for type-two fields.

§7. Heisenberg Fields and the Dynamical Map

In part I, section 5 we introduced Heisenberg fields '$\psi(x)$' which are dynamical and discussed the dynamical map. We now generalize these to finite temperature situations. Given the Heisenberg equation

$$\Lambda(\partial)\psi(x) = F[\psi(x)], \tag{7.1}$$

where

$$\frac{\partial}{\partial t}\psi(x) = i[H,\psi(x)] \tag{7.2}$$

we can obtain

$$\Lambda(\partial)^{*}\tilde{\psi}(x) = F[\psi(x)] = F^{*}[\tilde{\psi}^{\dagger}(x)]. \tag{7.3}$$

Using thermal doublet notation

$$\Lambda(\partial)\psi^{\alpha}(x) = P_{\alpha}F[\psi^{\alpha}(x)]. \tag{7.4}$$

Letting $\varphi^{o}(x)$ stand for a free physical field we have

$$\lambda(\partial)\varphi^{o}(x) = 0 \tag{7.5}$$

and

$$\Lambda(\partial)^{*}\tilde{\varphi}^{o}(x) = 0. \tag{7.6}$$

In part I we discussed the dynamical map of $\psi(x)$, we can write it in its fourier transformed form as

$$\psi(x) = \sum_{nm}\int dk_1\ldots dk_n\, dq_1\ldots dq_m\, F(x;k_1\ldots k_n;q_1\ldots q_m)$$
$$a^{\dagger}(k_1)\ldots a^{\dagger}(k_n)a(q_1)\ldots a(q_m). \tag{7.7}$$

and using our tilde axioms

$$\tilde{\psi}(x) = \sum_{nm}\int dh_1\ldots dk_n\, dq_1\ldots dq_m\, F^{*}(x;k_1\ldots k_n;q_1\ldots q_m)$$
$$\tilde{a}^{\dagger}(k_1)\ldots\tilde{a}^{\dagger}(k_n)\tilde{a}(q_1)\ldots\tilde{a}(q_m). \tag{7.8}$$

Using equations (3.33) and (3.34) for bosons and equations (3.22) and (3.24) for fermions we arrive at our temperature dependent dynamical map. The existence of such a mapping suggests that even at finite temperature $H = H_o$ and $\tilde{H} = \tilde{H}_o$. We write

$$\hat{H} \doteq \hat{H}_o, \tag{7.9}$$

where $$\hat{A} = A - \hat{A}^{\dagger}. \tag{7.10}$$

However, the fact that the tilde-particles have negative energies opens many decay channels, making many particles unstable. This makes us doubtful about use of asymptotic fields as physical fields. What is the complete set of physical particles at finite temperature is a very good question.

§8. The Kubo-Martin Schwinger Relation

Consider two operators A(t) and B(t) as being products of Heisenberg fields at common time 't'. Associating fermion numbers n_A and n_B to A(t) and B(t) respectively yields the following

$$\langle 0(\beta)|A(t)B(t')|0(\beta)\rangle$$
$$= (-1)^{(n_A-1)n_A/2}\langle 0(\beta)|\tilde{A}^\dagger(t + i\,\beta/2)B(t')|0(\beta)\rangle, \quad (8.1)$$

where we used the thermal state condition and the fact that the fermion number of $\tilde{A}^\dagger(t)$ is $-n_A$. Exchange of $\tilde{A}^\dagger(t + i\,\beta/2)$ and B(t') induces a sign factor of $(-1)^{n_A n_B}$ and hence we have

$$\langle 0(\beta)|A(t)B(t')|0(\beta)\rangle = (-1)^{(n_A-1)n_A/2}(-1)^{n_A n_B}$$
$$\langle 0(\beta)|B(t')\tilde{A}^\dagger(t + i\,\beta/2)|0(\beta)\rangle. \quad (8.2)$$

Application of the thermal state condition leads to

$$\langle 0(\beta)|B(t)A(t')|0(\beta)\rangle = (-1)^{(n_A-1)n_A/2}(-1)^{n_A n_B}(-1)^{(n_A+1)n_A/2}$$
$$\langle 0(\beta)|B(t')A(t)+i\beta)|0(\beta)\rangle \quad (8.3)$$
$$= (-1)^{n_A(n_A+n_B)}\langle 0(\beta)|B(t')A(t + i\beta)|0(\beta)\rangle$$
$$= \langle 0(\beta)|B(t')A(t + i\beta)|0(\beta)\rangle, \quad (8.4)$$

where we used the fact that equation (8.1) vanishes unless $n_A = -n_B$. Equation (8.4) is the Kubo-Martin-Schwinger relation, we see it follows straightforwardly from the thermal state condition.

§9. The L.S.Z. Formula at Finite Temperature

Let us now generalize the L.S.Z. formula, equation (I.10.24), to the case of finite temperature. Consider a real boson field $\psi(x)$ with

$$\varphi^{o}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k [u(k,x)a(k) + u^{*}(k,x)a^{\dagger}(k)] \quad (9.1)$$

where

$$u(k,x) = u(k)\exp(ikx - i\omega(k)t) \quad (9.2)$$

With equations (3.20) and (3.21) we have

$$a(k,\beta) = \int d^3k\, u(k,x)\overleftrightarrow{\partial}\{\cosh\theta(\beta,\partial)\varphi^{o}(x) - \sigma'\sinh\theta(\beta,\partial)\tilde{\varphi}^{o}(x)\} \quad (9.3)$$

$$\tilde{a}(k,\beta) = -\int d^3x\, u(k,x)\overleftrightarrow{\partial}\{\cosh\theta(\beta,\partial)\tilde{\varphi}^{o}(x) - \sigma'\sinh\theta(\beta,\partial)\delta^{o}(x)\} \quad (9.4)$$

where $\sinh\theta(\beta,\partial)$ and $\cosh\theta(\beta,\partial)$ are defined by equation (3.19) with

$$[\sinh\theta(\beta,\partial)]e^{ikx} = \sinh\theta(\beta,k) = \frac{e^{-\beta\omega(k)/2}}{1 - e^{\beta\omega(k)}} \quad (9.5)$$

Equations (9.3) and (9.4) show that we have interpolating fields

$$\psi(x,\beta) \equiv \cosh\theta(\beta,\partial)\{\psi(x) - \chi\} - \sigma'\sinh\theta(\beta,\partial)\{\tilde{\psi}(x) - \chi\} \quad (9.6)$$

for $a^{in}(k,\beta)$ and $a^{out}(k,\beta)$, and

$$\tilde{\psi}(x,\beta) \equiv \cosh\theta(\beta,\partial)\{\tilde{\psi}(x) - \chi\} - \sigma'\sinh\theta(\beta,\partial)\{\psi(x) - \chi\} \quad (9.7)$$

for $\tilde{a}^{in}(k,\beta)$ and $\tilde{a}^{out}(k,\beta)$. Here

$$\chi = \langle 0(\beta)|\psi|0(\beta)\rangle = \langle 0(\beta)|\tilde{\psi}|0(\beta)\rangle. \quad (9.8)$$

With (3.15) we note that

$$\int d^3x \{\varphi^o(x,\beta)\dagger^{\dagger} \psi(x,\beta) - \tilde{\varphi}^o(x,\beta)\dagger^{\dagger} \tilde{\psi}(x,\beta)\}$$
$$= \int d^3x\{\varphi^o(x), \dagger^{\dagger} (\psi(x) - \chi) - \tilde{\varphi}^o(x)\dagger^{\dagger} (\tilde{\psi}(x) - \chi)\} \tag{9.9}$$

where

$$\varphi^o(x,\beta) = \cosh \theta(\beta,\partial)\varphi^o(x) - \sigma' \sinh \theta(\beta,\partial)\tilde{\varphi}^o(x) \tag{9.10}$$

$$\tilde{\varphi}^o(x,\beta) = \cosh \theta(\beta,\partial)\tilde{\varphi}^o(x) - \sigma' \sinh \theta(\beta,\partial)\varphi^o(x), \tag{9.11}$$

are the physical fields corresponding to $\psi(x,\beta)$ and $\tilde{\psi}(x,\beta)$, respectively. With equation (9.9) we may follow the same arguments found in part I, §10 obtaining

$$S\psi(x) = \langle : T[\psi(x) \exp(-i \int d^4\xi\, J^o(\xi)\varphi(\xi) + \int d^4\xi\, \tilde{J}^o(\xi)\tilde{\varphi}(\xi)] : \rangle \tag{9.12}$$

$$S\tilde{\psi}(x) = \langle : T[\tilde{\psi}(x) \exp(-i \int d^4U\, J^o(\xi)\varphi(\xi) + i \int d^4(\xi)\tilde{J}^o(\xi)\tilde{\varphi}(\xi))] : \rangle \tag{9.13}$$

with

$$J^o(\xi) = \varphi^o\lambda(\partial), \tag{9.14}$$

$$J^{o\dagger}(\xi) = \varphi^{o\dagger}\lambda^*(\partial), \tag{9.14}$$

$$\varphi(\xi) = Z^{-1/2}[\psi(\xi) - \chi], \tag{9.15}$$

$$\tilde{\varphi}(\xi) = Z^{-1/2}[\tilde{\psi}(\xi) - \chi]. \tag{9.16}$$

where $\psi(x)$ could be a product of Heisenberg operators or taken as unity for an expansion of the S-matrix. When $\psi(x)$ is complex (fermion or boson) we obtain

$$S\,\psi(x) = \langle : T[\psi(x) \exp(-i \int d^4\xi (J^{o\dagger}(\xi)\varphi(\xi) + \varphi^{\dagger}(\xi)J^o(\xi))$$
$$+ i \int d^4\xi (\tilde{J}^{o\dagger}(\xi)\tilde{\varphi}(\xi) + \tilde{\varphi}^{\dagger}(\xi)\tilde{J}^o(\xi))] : \rangle \tag{9.17}$$

and its tilde conjugate, with

$$J^{o\dagger}(x) = \varphi^{o\dagger}(x)\lambda(\partial) \tag{9.18}$$

$$J^{o}(x) = \lambda(-\overleftarrow{\partial})\varphi^{o}(x) \tag{9.19}$$

$$\tilde{J}^{o\dagger}(x) = \tilde{\varphi}^{o\dagger}(x)\lambda^{*}(\partial), \tag{9.20}$$

$$\tilde{J}^{o}(x) = \lambda^{*}(-\overleftarrow{\partial})\tilde{\varphi}^{o}(x), \tag{9.21}$$

$$\varphi(x) = Z^{-1/2}[\psi(x) - \chi], \tag{9.22}$$

$$\tilde{\varphi}(x) = Z^{-1/2}[\tilde{\psi}(x) - \chi^{*}]. \tag{9.23}$$

We see that our dynamical map equation (9.17) now expresses $\psi(x)$ in terms of $\varphi^{o}(x)$ and $\tilde{\varphi}^{o}(x)$. Using the thermal state condition one can make it depend on $\varphi^{o}(x)$ only.

Here we end our discussion of equilibrium TFD. Much insight was gained about it when TFD was generalized to accommodate non-equilibrium situations. This will be seen in part III. This discussion of TFD is by no means complete; symmetry breaking and the renormalization group have been formulated at finite temperature, and better understood, using TFD.

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Part III: Thermal Field Dynamics for Nonequilibrium Situations

§0. Introduction

Let us understand the state of non-equilibrium many body phenomena before the advent of TFD. With every macroscopic dynamical variable of some sub-system, which is coupled to a reservoir we associate an operator average $\langle a(t)\rangle$ which is defined as

$$\langle a(t)\rangle \equiv \frac{Tr[W(t_o)a(t)]}{Tr[W(t_o)]} . \tag{0.1}$$

Density matrix, Green's function and quantum noise operator theories represent the three different approaches to the calculation of such operator moments and correlation functions, ie. $\langle a(t)\rangle$ and $\langle a^\dagger(t')a(t)\rangle$. Given a time independent Hamiltonian we have

$$\frac{\partial}{\partial t} a(t) = i[H,a(t)]_- \tag{0.2}$$

where $H = H_s+H_r+H_{sr}$ is the total Hamiltonian. The Hamiltonian for the subsystem which is to some extent characterized by the dynamical variable $\langle a(t)\rangle$ is 'H_s'. The Hamiltonian of the reservoir is 'H_r'. The subsystem-reservoir interaction Hamiltonian is 'H_{sr}'. Equation (0.2) can be solved formally as

$$a(t) = u^\dagger(t,t_o)a(t_o)u(t,t_o) \tag{0.3}$$

$$u(t,t_o) = \exp(-iH(t-t_o)) \tag{0.4}$$

$$u(t_o,t_o) = 1. \tag{0.5}$$

When equation (0.3) is substituted into equation (0.1) we have

$$\langle a(t)\rangle = \frac{\mathrm{Tr}[u^{\dagger}(t,t_o)a(t_o)u(t,t_o)W(t_o)]}{\mathrm{Tr}[W(t_o)]} \tag{0.6}$$

$$= \frac{\mathrm{Tr}[a(t_o)u(t,t_o)W(t_o)u^{\dagger}(t,t_o)]}{\mathrm{Tr}[W(t_o)]} \tag{0.7}$$

$$\equiv \frac{\mathrm{Tr}[a(t_o)W(t)]}{\mathrm{Tr}(W(t_o)]} \tag{0.8}$$

where we have defined

$$W(t) = u(t,t\ )W(t\ )u^{\dagger}(t,t_o), \tag{0.9}$$

and hence W(t) satisfies the Liouville equation

$$\frac{\partial}{\partial t} W(t) = -i[H,W(t)]_{-} \tag{0.10}$$

At this point comes the parting of the ways between the three above mentioned theoretical approaches. Density matrix theory tackles, in some way, equation (0.10). With a solution for 'W(t)' one obtains average values of any operator or correlation via equation (0.8). The Green's function approach emphasizes '$u(t,t_o)$' via equation (0.6). '$u(t,t_o)$' can be made a functional of some external time varying forces which are coupled to the appropriate subsystem's operator. Then by varying these fictitious external forces one finds the subsystems response is obtainable. Lastly quantum noise operator theory deals with the subsystem-reservoir interaction on an operator level. Equation (0.2) is cast into a quantum noise operator equation of motion and solved in some way.

As successful as these three theories have been they can not accommodate the vast wealth of theoretical calculation methods which quantum field theory has to offer for the simple reason that they do not define a system's state. Once one has a consistent and well defined concept of a state one can define excitations of it which are realizable modes of a system and all moments and correlations are available. In part II we saw how $|0(\beta)>$ and excitations of it were just such a state for equilibrium situations. We now introduce TFD for non-equilibrium situations[1], which is based upon a similar but more general concept of the thermal state.

§1. The Liouville Space and Superoperators

As it was necessary to make the abstraction of a physical particle state to a vector in a separable Hilbert space we now find it necessary to make a further abstraction. The density matrix 'W(t)' of a system contains a complete description of that system at time 't'. The equation of motion for 'W(t)' is the Liouville equation

$$i \frac{\partial}{\partial t} W(t) = [H,W(t)]_- . \qquad (1.1)$$

It is to our advantage to seek a "Schrodinger type" equation of motion for W(t) and this is accomplished with the use of superoperators and supervectors. With this realization we can develop a quantum field theory around these supervector states. We now introduce this next level of abstraction. The thermal Liouville space.

After giving the general properties of the Liouville space and introducing superoperators, we build a linear vector space which will be called the thermal-Liouville space.

The Liouville space[2,3] can be spanned by a complete orthonormal basis

$$|mn\rangle\rangle = ||m\rangle\langle n|\rangle\rangle, \tag{1.2}$$

$$\langle\langle mn| = |nm\rangle\rangle^{\dagger} = \langle\langle(|m\rangle\langle n|)^{\dagger}| = \langle\langle|n\rangle\langle m||, \tag{1.3}$$

which satisfies

$$\langle\langle mn|m'n'\rangle\rangle = \delta_{mm'}\delta_{nn'} \tag{1.4}$$

$$\sum_{mn} |mn\rangle\rangle\langle\langle mn| = \hat{1}, \tag{1.5}$$

where $\{|n\rangle = |n_1,n_2,\ldots\rangle\}$ is a complete orthonormal basis of the Fock space of Part II, which is generated by cyclic operation of the creation operators $a_i^{\dagger}$ on the vacuum $|0\rangle$. Let A consist of a_i and $a_i^{\dagger}$, define

$$|A\rangle\rangle = \sum_{mn} |mn\rangle\rangle\langle m|A|n\rangle, \tag{1.6}$$

$$\langle\langle A| = \sum_{mn} \langle n|A|m\rangle\langle\langle mn|. \tag{1.7}$$

We then have

$$\langle\langle mn|A\rangle\rangle = \langle m|A|n\rangle, \tag{1.8}$$

$$\langle\langle A|mn\rangle\rangle = \langle n|A|m\rangle. \tag{1.9}$$

$|A\rangle\rangle$ and $\langle\langle A|$ then belong to the Liouville space. They are related to each other through

$$\langle\langle A| = |A^\dagger\rangle\rangle^\dagger. \tag{1.10}$$

We have also

$$\langle\langle A|B\rangle\rangle = \sum_n \langle n|AB|n\rangle = \mathrm{Tr}\ AB. \tag{1.11}$$

For $A = \hat{1}$ we have from equations (1.6) and (1.7)

$$|1\rangle\rangle = \sum_n |nn\rangle\rangle, \tag{1.12}$$

$$\langle\langle 1| = \sum_n \langle\langle nn|. \tag{1.13}$$

The operators which induce linear transformations among these vectors (elements of the Liouville space) are called superoperators.[4,5]

Following Schmutz, reference 5, let us define the set of superoperators a_i and $\tilde{a}_i$ by

$$a_i|mn\rangle\rangle = |a_i|m\rangle\langle n|\rangle\rangle, \tag{1.14}$$

$$\tilde{a}_i|mn\rangle\rangle = \sigma^{\mu+1}||m\rangle\langle n|a_i^\dagger\rangle\rangle, \tag{1.15}$$

with $\mu = \sum_i (m_i - n_i)$, where we have used the same notation for superoperators and ordinary operators. No confusion should result, for a superoperator acts on supervectors $|x\rangle\rangle$ an ordinary operator acts on vectors $|x\rangle$. Examining the matrix element of $\tilde{a}^\dagger$ we find

$$\begin{aligned}\langle\langle m'n'|\tilde{a}_i^\dagger|nm\rangle\rangle &= \langle\langle mn|\tilde{a}_i|m'n'\rangle\rangle^* \\ &= \sigma^{\mu'+1}\langle\langle mn||m'\rangle\langle n'|a_i^\dagger\rangle\rangle^* \\ &= \sigma^{\mu'+1}\langle m|m'\rangle^*\langle n'|a_i^\dagger|n\rangle^* \\ &= \sigma^{\mu+1}\langle m'|m\rangle\langle n|a_i|n'\rangle \\ &= \sigma^{\mu}\langle\langle m'n'||m\rangle\langle n|a_i\rangle\rangle. \end{aligned} \tag{1.16}$$

where $\mu' = \sum_i (m'_i - n'_i)$ and hence we find

$$\tilde{a}_i^\dagger |mn\rangle\rangle = \sigma^\mu ||m\rangle\langle n|a_i\rangle\rangle, \tag{1.17}$$

similarly

$$a_i^\dagger |mn\rangle\rangle = |a_i^\dagger |m\rangle\langle n|\rangle\rangle. \tag{1.18}$$

The four superoperators $(a_i, a_i^\dagger, \tilde{a}_i, \tilde{a}_i^\dagger)$ form a basic set of superoperators and hence any superoperator is a linear sum of products of them.

From equations (1.14), (1.15), (1.17) and (1.18) follow the commutation relations among the superoperators, i.e.,

$$[a_i, a_j^\dagger]_\sigma = [\tilde{a}_i, \tilde{a}_j^\dagger]_\sigma = \delta_{ij}, \text{ all others zero.} \tag{1.19}$$

We also obtain

$$a_i^\dagger a_i |mn\rangle\rangle = m_i |mn\rangle\rangle, \tag{1.20}$$

$$\tilde{a}_i^\dagger \tilde{a}_i |mn\rangle\rangle = n_i |mn\rangle\rangle, \tag{1.21}$$

and

$$a_i |00\rangle\rangle = \tilde{a}_i |00\rangle\rangle = 0, \tag{1.22}$$

where

$$|00\rangle\rangle = ||0\rangle\langle 0|\rangle\rangle, \tag{1.23}$$

which is called the supervacuum.[4]

From (1.13), we have

$$|1\rangle\rangle = \sum_n |nn\rangle\rangle = \exp\left(\sum_i a_i^\dagger \tilde{a}_i^\dagger\right)|00\rangle\rangle, \tag{1.24}$$

$$\langle\langle 1| = \sum_n \langle\langle nn| = \langle\langle 00|\exp\left(\sum_i \tilde{a}_i a_i\right). \tag{1.25}$$

It follows from equations (1.24) and (1.25) that

$$a_i|1\rangle\rangle = \tilde{a}_i^\dagger|1\rangle\rangle, \tag{1.26}$$

$$a_i^\dagger|1\rangle\rangle = \sigma\tilde{a}_i|1\rangle\rangle. \tag{1.27}$$

According equation (1.3) we have

$$\langle\langle mn|a_i^\dagger = \langle\langle|n\rangle\langle m|a_i^\dagger|, \tag{1.28}$$

$$\langle\langle mn|\tilde{a}_i^\dagger = \langle\langle a_i|n\rangle\langle m||\sigma^{\mu+1}, \tag{1.29}$$

$$\langle\langle mn|a_i = \langle\langle|n\rangle\langle m|a_i|, \tag{1.30}$$

$$\langle\langle mn|\tilde{a}_i = \langle\langle a_i^\dagger|n\rangle\langle m||\sigma^{\mu}, \tag{1.31}$$

and

$$\langle\langle 1|a_i^\dagger = \langle\langle 1|\tilde{a}_i, \tag{1.32}$$

$$\langle\langle 1|a_i = \langle\langle 1|\tilde{a}_i^\dagger\sigma. \tag{1.33}$$

From equations (1.6), (1.14) and (1.15) it follows that when A and B consist only of a and $a^\dagger$,

$$A|1\rangle\rangle = |A\rangle\rangle,$$

$$BA|1\rangle\rangle = B|A\rangle\rangle = |BA\rangle\rangle, \tag{1.34}$$

and from equations (1.17), (1.28) and (1.29) we have

$$\langle\langle 1|A = \langle\langle A|, \tag{1.35}$$

$$\langle\langle 1|AB = \langle\langle A|B = \langle\langle AB|. \tag{1.36}$$

These relations satisfy the self-consistency condition

$$\langle\langle A|B|C\rangle\rangle = \langle\langle A|BC\rangle\rangle = \langle\langle AB|C\rangle\rangle. \tag{1.37}$$

Then we also have, using (1.11)

$$\langle\langle 1|A\rangle\rangle = \langle\langle A|1\rangle\rangle = \langle\langle 1|A|1\rangle\rangle = \mathrm{Tr}A. \tag{1.38}$$

We now set up the rule for the tilde conjugation. We first note that (1.15) gives

$$c(\tilde{a}_i^\dagger \ldots \tilde{a}_j^\dagger)(\tilde{a}_k \ldots \tilde{a}_\ell)|mn>>$$
$$= \bar{\sigma}||m><n|[c^*(a_i^\dagger \ldots a_j^\dagger)(a_k \ldots a_\ell)]^\dagger>>, \qquad (1.39)$$

where c is a complex c-number and $\bar{\sigma}$ is the product of $\sigma^{\mu+1}$ which was given in (1.15). This relation indicates that we are dealing with the same tilde conjugation as was introduced in part II, namely;

$$(AB)^{\sim} = \tilde{A}\tilde{B}, \qquad (1.40)$$

$$(c_1 A + c_2 B)^{\sim} = c_1^* \tilde{A} + c_2^* \tilde{B}. \qquad (1.41)$$

For an arbitrary operator A consisting of a and $a^\dagger$ we have

$$\tilde{A}|mn>> = \bar{\sigma}||m><n|A^\dagger>>. \qquad (1.42)$$

When A has the form $c(a_{i_1}^\dagger \ldots a_{i_m}^\dagger)(a_{k_1} \ldots a_{k_n})$ with a c-number c, (1.27) leads to, as discussed in part II section 4;

$$A|1>> = \sigma^{F(F+1)/2}\tilde{A}^\dagger|1>>, \qquad (1.43)$$

$$A^\dagger|1>> = \sigma^{F(F-1)/2}A|1>>, \qquad (1.44)$$

where $F \equiv m-n$. Here it was considered that the total number of permutations needed in reversing the order of the operator elements in A is $(m+n)(m+n-1)/2$ which is equal to [F(F+1)/2 + m + an even number], and that each $a^\dagger$ in A contributes to the phase factor on the right hand side of equation (1.43) by an amount σ, according to equation (1.27). In a similar manner, equations (1.32) and (1.33) lead to

$$<<1|A^\dagger = <<1|\tilde{A}\ \sigma^{F(F+1)/2}, \qquad (1.45)$$

$$<<1|A = <<1|\tilde{A}^\dagger\ \sigma^{F(F-1)/2}. \qquad (1.46)$$

Now (1.34) gives

$$|BA>> = BA|>> = \sigma^{F(F+1)/2}{}_{B}\tilde{A}^{\dagger}|1>>$$
$$= \sigma^{F_A(F_A+1)/2+F_AF_B}{}_{B}\tilde{A}^{\dagger}|B>>, \qquad (1.47)$$

where $\sigma^{F_AF_B}$ is created by the commutation among B and $\tilde{A}^{\dagger}$. Here F_B is related to B in the same way as F_A to A. Thus F_A and F_B are the fermion numbers of A and B, respectively. The relations (1.43) through (1.47) hold true even if A is the linear sum of the products of the above form with common m-n and when B has the same structure, because, then F_A and F_b can be assigned to A and B, respectively. When B has an inverse, (1.47) gives

$$BAB^{-1}|B>> = \sigma^{F_A(F_A+1)/2+F_AF_B}{}_{B}\tilde{A}^{\dagger}|B>>. \qquad (1.48)$$

This is the most general form of the thermal state condition. We will discuss this shortly.

Finally, we note that by comparing the tilde conjugate of the both sides of (1.43) [or (1.45) with (1.44)] (1.46) gives

$$\tilde{\tilde{A}} = \sigma_A A, \qquad (1.49)$$

where σ_A is the fermion number of A, i.e., $\sigma_A = \sigma^{F_A}$. A similar argument holds for A consisting of $\tilde{a}$ and $\tilde{a}^{\dagger}$. Note that we can modify[6],[7] the phase factor in the definition of the superoperators in (1.14) in such a manner that $\tilde{\tilde{A}} = A$.

However, in this paper we use (1.14) which leads to (1.49). This is very similar to the arguments of part IV for non-super operators.

As particular cases for (1.49), we have

$$\tilde{a}_i = \sigma a_i, \quad \tilde{a}_i^\dagger = \sigma a_i^\dagger. \tag{1.50}$$

With this understanding of supervectors and superoperators we now introduce the linear vector space in which TFD for non-equilibrium situations is based, called the thermal-Liouville space. We assume that each thermal state is represented by a vector in the Liouville space. In equilibrium TFD we took as our vacuum the ground state of a system and developed an annihilation (creation) operator which annihilated (excited) this vacuum '$|0(\beta)>$'. In TFD for non-equilibrium situations we do a similar thing; the density operator which describes a system W(t) is "made" a supervector and called the thermal-vacuum ket-vector.

$$W(t)|1>> = |W(t)>> = \text{thermal-vacuum ket-vector.} \tag{1.51}$$

We understand that in equation (1.51) the 'W(t)' appearing on the left hand side is a superoperator while the '$|W(t)>>$' appearing on the right hand side is a supervector. The operator 'W(t)' contained within this ket-supervector is then an ordinary operator realized in our physical Fock-space obtained in a self consistent way as a subset of a separable Hilbert space. Clearly for equilibrium situations, i.e. $W = e^{-\beta H}$, equation (1.51) agrees with our definition of

'$|0(\beta)>$' found in part II. The superoperators a and $\tilde{a}^\dagger$ act within the ket as annihilation operators (as shown on equations (1.14) and (1.29)) but they do not annihilate the thermal-vacuum ket-vector. We then have the task, as we did in equilibrium TFD, of finding the quasi-particle superoperators which annihilate the thermal vacuum. We will do this shortly. Let us finish this section by showing that the goal of writing equation (1.1) as a Schrodinger-type equation can now be realized. Let us multiply equation (1.1) by $|1>>$,

$$i \frac{\partial}{\partial t} W(t)|1>> = [H,W(t)]_-|1>> \tag{1.52}$$

$$i \frac{\partial}{\partial t} |W(t)>> = H\, W(t)|1>> - W(t)H|1>> \tag{1.53}$$

Using equation (1.47) and the fact that our Hamiltonian is bosonic we have

$$i \frac{\partial}{\partial t} |W(t)>> = H\, W(t)|1>> - \tilde{H}^\dagger W(t)|1>> \tag{1.54}$$

$$i \frac{\partial}{\partial t} |W(t)>> = H - \tilde{H}^\dagger |W(t)>> \tag{1.55}$$

$$i \frac{\partial}{\partial t} |W(t)>> = \hat{H}|W(t)>>, \tag{1.56}$$

where we have retained the same definition for '$\hat{}$' for superoperators as we had earlier for ordinary operators. Equation (1.56) is the Schrodinger form of the Liouville equation of motion for 'W(t)'. We see that the time evolution of the thermal vacuum is generated by the superoperator $\hat{H}$, the full super-Hamiltonian.

§2. Heisenberg and Interaction Representations

The true advantage of getting equation (1.56) is that we can do that which is done in ordinary quantum field theory, namely develop other representations by shifting the time evolution from the state to the operators either completely (Heisenberg picture) or partially (Interaction picture).

Solving equation (1.56) formally we obtain

$$|W(t)>> = \hat{S}(t-s)|W(s)>>, \tag{2.1}$$

where

$$\hat{S}(t) = \exp[-i\hat{H}t]. \tag{2.2}$$

Note that we did not require that $\hat{H}$ be Hermitian. Thus $\hat{S}$ is not necessarily unitary.

The thermal average is given by $<<1|A|W(t)>>$ when $|W(t)>>$ is normalized according to

$$<<1|W(t)>> = 1, \tag{2.3}$$

then

$$<<1|A|W(t)>> = <<1|A\ W(t)|1>> = \sum_n <n|A\ W(t)|n> \tag{2.4}$$

$$= <A> \text{ of statistical mechanics.} \tag{2.5}$$

In order that all the axioms of this more general TFD may be presented all at once we take $<<1|\hat{H} = 0$, and will continue. Discussion about this will be contained in section 4. We then have

$$<<1|\hat{A}|W(t)>> = <<1|\hat{A}(t)|W(t_o)>>, \tag{2.6}$$

where

$$\hat{A}(t) = \hat{S}^{-1}(t-t_o)\hat{A}\,\hat{S}(t-t_o). \tag{2.7}$$

The superoperator $\hat{A}(t)$ will be called the Heisenberg representation of the superoperator $\hat{A}$.

The state $|W(t_o)>>$ is called the thermal vacuum ket-vector in the Heisenberg representation. This state is determined by the initial condition for the system at $t = t_o$ (ie., by the experimental setup of the system at the initial time t_o).

The Heisenberg equation of motion for the superoperators is

$$\partial_t \hat{A}(t) = i[\hat{H}, \hat{A}(t)]. \tag{2.8}$$

As particular cases of (2.7), we have

$$a(t) = \hat{S}^{-1}(t-t_o)a\,\hat{S}(t-t_o), \tag{2.9}$$

$$a^{\dagger\dagger}(t) = \hat{S}^{-1}(t-t_o)a^{\dagger}\,\hat{S}(t-t_o), \tag{2.10}$$

$$\tilde{a}(t) = \hat{S}^{-1}(t-t_o)\tilde{a}\,\hat{S}(t-t_o), \tag{2.11}$$

$$\tilde{a}^{\dagger\dagger}(t) = \hat{S}^{-1}(t-t_o)\tilde{a}^{\dagger}\,\hat{S}(t-t_o). \tag{2.12}$$

It should be noted that $a^{\dagger\dagger}(t)$ and $\tilde{a}^{\dagger\dagger}(t)$ are not Hermitian conjugation to $a(t)$ and $\tilde{a}(t)$, respectively, when $\hat{S}$ is not unitary, although they satisfy the "canonical" relations:

$$[a_i(t), a_j^{\dagger\dagger}(t)]_\sigma = [\tilde{a}_i(t), \tilde{a}_j^{\dagger\dagger}(t)]_\sigma = \delta_{ij}. \tag{2.13}$$

The basic bra-vectors are created by cyclic actions of annihilation operators on the thermal vacuum bra-vector which is $\langle\langle 1|$.

In order for equations (2.11) or (2.12) to follow from equations (2.9) or (2.10) respectively, through the tilde conjugation rule, we find a very fundamental principle for $\hat{H}$; that is, $\hat{H}$ should satisfy the condition $[i\hat{H}]^{\sim} = i\hat{H}$. When an operator $\hat{A}$ satisfies $[i\hat{A}]^{\sim} = i\hat{A}$, $\hat{A}$ is said to be Tildian. Thus $\hat{H}$ should always be Tildian, though it does not necessarily have to be Hermitian. Thus the Tildian property of $\hat{H}$ is more fundamental than the Hermitian property.

Thus $\langle\langle 1|$ and $|W(t_o)\rangle\rangle$ are considered to be thermal vacuum states (in the Heisenberg representation) for the bra- and ket-vectors, respectively.

The interaction representation comes about, as in part I, by shifting only the time evolution of the free system to the operators ie.

$$\langle\langle 1|\hat{A}|W(t)\rangle\rangle$$

$$= \langle\langle 1|\hat{S}_o(t-t_o)\hat{S}_o^{-1}(t-t_o)\hat{A}\,\hat{S}_o(t-t_o)\hat{S}_o^{-1}(t-t_o)|W(t)\rangle\rangle \tag{2.14}$$

$$= \langle\langle 1|\hat{A}_i(t-t_o)|W(t-t_o)\rangle\rangle_I \tag{2.15}$$

where

$$\hat{S}_o(t) = \exp(-i\,\hat{H}_o t) \tag{2.16}$$

and hence

$$|W(t-t_o)>>_I = \exp(i\hat{H}_o(t-t_o))|W(t)>> \qquad (2.17)$$

$$= \exp(i\hat{H}_o(t-t_o))\exp(-i\hat{H}(t-t_o))|W(t_o)>>$$

$$\equiv T\exp(-i\int_{t_o}^{t}\hat{S}_o^{-1}(t'-t_o)\hat{H}_{int}\hat{S}_o(t'-t_o)dt')|W(t_o)>> \qquad (2.18)$$

The state $|W(t)>>_I$ is called the thermal vacuum ket-vector in the interaction representation.

It is worthwhile to note here that there is a space similar to the thermal-Liouville space, which will be called the mirror space of the thermal-Liouville space, the bra and ket state vectors of which are constructed on the mirror thermal vacuum states $<<W(t_o)|$ and $|1>>$, respectively. In the mirror space, we can define the Heisenberg representation of the mirror superoperator which is nothing but the mirror operator introduced in ref. 8. To date no symmetric formalism exists for non-equilibrium TFD; further investigation of the mirror Liouville space may however yield just such a formalism.

§3. The Thermal State Condition and Quasi-Particle Operators

Recalling equation (1.48), we let $B = W(t_o)$ and $A = a_k(t_o)$, and hence we have

$$W(t_o)a_k(t_o)W^{-1}(t_o)|W(t_o)>> = \tilde{a}_k^{\dagger}(t_o)|W(t_o)>>, \quad (3.1)$$

where the subscript 'k' describes the wave number. If we have an equilibrium system[9], equation (3.1) leads to

$$\exp(-\beta\hat{H})a_k \exp(\beta\hat{H})|W>> = \tilde{a}_k^{\dagger}|W>>, \quad (3.2)$$

Equation (3.2) in turn leads to

$$e^{\beta\omega_k}a_k|W>> = \tilde{a}_k^{\dagger}|W>> \quad (3.3)$$

or

$$a_k|W>> = e^{-\beta\omega}\tilde{a}_k^{\dagger}|W>>, \quad (3.4)$$

which is just the equilibrium thermal state condition previously derived. If we are considering a non-equilibrium system the density matrix can not be a stationary one and hence we expect a more general form for our thermal state condition, ie.

$$a_k|W>> = f_k\,\tilde{a}_k^{\dagger}|W>> + (\text{higher order non-linear terms in } \tilde{a},\tilde{a}^{\dagger},a,a^{\dagger})|W>> \quad (3.5)$$

We treat the non-linear terms in equation (3.5) as perturbations and take as our unperturbed non-equilibrium thermal state condition

$$a_k|W>> = f_k\tilde{a}_k^{\dagger}|W>>. \quad (3.6)$$

As in usual field theory we now treat any interaction term contained in the Hamiltonian as a perturbative effect and to zeroth order approximation we have

$$a_k(t) = \hat{S}_o^{-1}(t-t_o)a_k\ \hat{S}_o(t-t_o) \tag{3.7}$$

$$\tilde{a}_k^{\dagger\dagger}(t) = \hat{S}_o^{-1}(t-t_o)\tilde{a}_k^{\dagger}\ \hat{S}_o(t-t_o). \tag{3.8}$$

Since $\hat{H}_o$ contained in $\hat{S}_o^{-1}(t)$ is of a bilinear form, the thermal state condition at time t in the unperturbed Heisenberg representation or zeroth order interaction perturbation should be linear in $a_k(t)$ and $\tilde{a}_k^{\dagger}(t)$ as it was at '$t=t_o$', and '$t=\infty$'. Hence we write using equations (3.6), (3.7) and (3.8),

$$a_k(t)|W(t_o)>> = f_k(t-t_o)\tilde{a}_k^{\dagger\dagger}(t)|W(t_o)>> \tag{3.9}$$

and

$$\tilde{a}_k(t)|W(t_o)>> = \sigma\ f_k(t-t_o)a_k^{\dagger\dagger}(t)|W(t_o)>>. \tag{3.10}$$

Considering equations (1.32) and (1.33) we have

$$<<1|a_k^{\dagger}(t) = <<1|\tilde{a}_k(t) \tag{3.11}$$

$$<<1|a_k(t) = <<1|\tilde{a}_k^{\dagger}(t)\sigma. \tag{3.12}$$

These four equations (3.9), (3.10), (3.11) and (3.12) constitute collectively the unperturbed thermal state condition.

We define the annihilation and creation quasi-particle superoperators by

$$\gamma(t) = z^{1/2}(t-t_o)[a(t) - f(t-t_o)\tilde{a}^{\dagger\dagger}(t)], \tag{3.13}$$

$$\tilde{\gamma}^{\dagger}(t) = Z^{1/2}(t-t_o)[\tilde{a}^{\dagger\dagger}(t) - \sigma a(t)], \qquad (3.14)$$

respectively because, then, due to our thermal state condition

$$\gamma(t)|W(t_o)\rangle\rangle = 0, \quad \langle\langle 1|\tilde{\gamma}^{\dagger}(t) = 0. \qquad (3.15)$$

The tilde conjugation of (3.15) leads to

$$\tilde{\gamma}(t)|W(t_o)\rangle\rangle = 0, \quad \langle\langle 1|\gamma^{\dagger}(t) = 0. \qquad (3.16)$$

The normalization factor $Z^{1/2}(t)$ is determined by the canonical commutation relation

$$[\gamma(t),\gamma^{\dagger}(t)]_{\sigma} = [\tilde{\gamma}(t),\tilde{\gamma}^{\dagger}(t)]_{\sigma} = 1, \qquad (3.17)$$

while the other commutation relations vanish. The result is

$$Z(t) = 1 + n_{\sigma}(t), \qquad (3.18)$$

where

$$n_{\sigma}(t) = f_{\sigma}(t)/[1-f_{\sigma}(t)], \qquad (3.19)$$

with

$$n_{\sigma}(t) = \sigma n(t), \qquad (3.20)$$

$$f_{\sigma}(t) = \sigma f(t). \qquad (3.21)$$

Using the relations (3.13), (3.14), (3.15) and (3.16), we obtain

$$n(t-t_o) = \langle\langle 1|a^{\dagger\dagger}(t)a(t)|W(t_o)\rangle\rangle. \qquad (3.22)$$

The above argument shows one of the most significant roles played by the thermal state condition; the latter condition specifies the thermal vacuum and creation and annihilation superoperators for the quasi-particles. Note

that equation (3.22) evaluated at $t=t_o$ and $t=\infty$ gives us $f(t_o)$ and $f(\infty)$ via equation (3.19).

Note that although we have used the same notations $a(t)$, $\tilde{a}^{\dagger\dagger}(t)$ and $|W(t_o)>>$ both for the Heisenberg and the interaction representation, we expect that one can distinguish between them by the context. This point about our notation is not to be considered a problem which need be corrected - it is more correctly evidence of an important aspect of TFD. We have two theoretical specifications for a system, the Hamiltonian and the thermal state condition, <u>each can accommodate perturbation theory</u>. Whenever we have an operator in TFD, say '$\hat{A}(t)$' it contains two different possible approximations. The approximation in correlations given by the level of perturbation used in the thermal state condition and operator 'A(t)' contains an approximation in its time development given by the level of perturbation used to determine its time evolution generator. At present we are working with a zeroth order perturbation of the thermal state condition, but thus far we have not chosen to what perturbation order we will take the time evolution operator $\hat{S}(t)$.

We now define the thermal-Liouville space in which TFD for non-equilibrium situations is constructed. The thermal-Liouville space is nothing but the linear vector space spanned by the set of bra and ket vectors which are

generated, respectively, by cyclic operations of the annihilation superoperators $\gamma(t)$ and $\tilde{\gamma}(t)$ on the thermal vacuum $\langle\langle 1|$, and of the creation superoperators $\gamma^{\varsigma}(t)$ and $\tilde{\gamma}^{\varsigma}(t)$ on the thermal vacuum $|W(t_o)\rangle\rangle$.

Again let us state that both the deviation of $\hat{H}$ from the unperturbed Hamiltonian $\hat{H}_o$ and the deviation of the thermal state condition from its unperturbed linear form [i.e., equations (3.11) and (3.12)] are considered as perturbative effects. By adopting the usual definition of the normal product for the quasi-particle superoperators [i.e., when a product has a form in which all the creation superoperators (γ^{ς} and $\tilde{\gamma}^{\varsigma}$) stand to the left of the annihilation superoperators (γ and $\tilde{\gamma}$), it is called a normal product], we obtain a Wick-type formula for non-equilibrium TFD. This Wick-type formula should lead us to Feynman-type diagrams for multi-time Green's functions in the interaction representation. We then obtain a Feynman-type diagram method for perturbative calculations for non-equilibrium TFD when a perturbative interaction is introduced in $\hat{H}$. Calculations exemplifying this field theoretical strength of non-equilibrium TFD are now in progress. We can also formulate the generating functional method in non-equilibrium TFD, this will be shown in section 6.

Note that the perturbational calculation leads us to an expression of the Heisenberg superoperators in terms of product of quasi-particle superoperators. This is an extension of the concept of the dynamical map in the usual quantum field theory to TFD, for non-equilibrium situations.

Historically $\hat{H}$ was constructed by the elimination of the reservoir degrees of freedom.[1] The system's effective Hamiltonian was obtained by taking the reservoir's vacuum expectation of the complete Hamiltonian, leaving the free systems Hamiltonian with a shift in the free energy and a dissipative term coming from the system-reservoir interaction.

According to refs. 10 and 11, the entropy for the nonequilibrium state in the thermal-Liouville space is given by

$$S(t) = -\frac{1}{2}\,\ell n\, \Omega(t)\Omega(\infty), \tag{3.18}$$

with

$$\begin{aligned}\Omega(t) &= \langle\langle 1|W^{\dagger}(t)|W(t)\rangle\rangle \\ &= \langle\langle W^{\dagger}(t)|W(t)\rangle\rangle. \end{aligned} \tag{3.19}$$

The time derivatives of $\Omega(t)$ are given by

$$d_t\Omega(t) = -\langle\langle W^{\dagger}(t)|[(i\hat{H})^{\dagger} + i\hat{H}]|W(t)\rangle\rangle, \tag{3.20}$$

$$\begin{aligned} d_t^2\Omega(t) &= \langle\langle W^{\dagger}(t)|[(i\hat{H})^{\dagger} + i\hat{H}]^2|W(t)\rangle\rangle \\ &\quad - \langle\langle W^{\dagger}(t)|[i\hat{H},(i\hat{H})^{\dagger}]|W(t)\rangle\rangle \end{aligned} \tag{3.21}$$

The sign of $d_t\Omega(t)$ should be determined by the boundary

condition of the system (i.e., if the system is open or closed etc.). Note that if $\hat{H}$ is Hermitian $\Omega(t)$ remains constant in time.

§4. The Axioms for the Generalization of T.F.D. to Non-Equilibrium Situations (12)

With this formalism established, we can reformulate everything in terms of usual field theory language. Leaving behind the derivations for all the preceding formulas, we can replace this formalism with a small number of axioms, expressed in terms of operators, not superoperators, and vectors, not supervectors, and contain it in a thermal space, not the thermal Liouville space. With these axioms one finds no mention of a reservoir (remarkable). This is very appealing; because the "need" for a reservoir by all other thermal theories does not seem to be supported by nature.

It is now appropriate to list the seven axioms regarding operators and vectors which when included with the original three axioms of TFD (part II) and the axioms of quantum field theory (part I), yield a complete quantum field theory for situations which include non-equilibrium situations.

A1. The equation of motion for the thermal vacuum ket-vector in the Schrodinger representation $|W(t)>>$ is given by

$$\partial_t |W(t)>> = -i\hat{H}|W(t)>>, \tag{4.1}$$

where $\hat{H}$ is an operator consisting of a, $a^\dagger$, $\tilde{a}$ and $\tilde{a}^\dagger$. The equation (2.1) will be called the Schrodinger equation or the master equation.

A2. The superoperator $\hat{H}$ should satisfy the relation

$$(i\hat{H})^{\sim} = i\hat{H}, \text{ ie. } \hat{H} \text{ is Tildian.} \tag{4.2}$$

A3. The requirement of the conservation of the inner-product between the thermal vacuum bra-vector $\langle\langle 1|$ and the thermal vacuum ket-vector $|W(t)\rangle\rangle$ reads

$$\langle\langle 1|\hat{H} = 0. \tag{4.3}$$

A4. The thermal state condition for the thermal vacuum bra-vector is given by

$$\langle\langle 1|a^{\dagger} = \langle\langle 1|\tilde{a}, \tag{4.4}$$

$$\langle\langle 1|a = \langle\langle 1|\tilde{a}^{\dagger}\sigma. \tag{4.5}$$

A5. The requirement of the existence of the stationary thermal vacuum ket-vector in the Schrodinger representation reads

$$\tilde{H}|W(\infty)\rangle\rangle = 0. \tag{4.6}$$

A6. The thermal state condition for the stationary thermal vacuum ket-vector $|W(\infty)\rangle\rangle$. The unperturbed

part of the thermal state condition is given in the form

$$a|W(\infty)>> = \bar{f}\tilde{a}^{\dagger}|W(\infty)>>, \tag{4.7}$$

$$\tilde{a}|W(\infty)>> = \sigma\bar{f}a^{\dagger}|W(\infty)>>, \tag{4.8}$$

where $\bar{f}$ is some c-number function. From equation (3.11) we see $\bar{f} = f_k(\infty)$.

A7. The thermal state condition for the thermal vacuum ket-vector in the Heisenberg representation $|W(t_o)>>$. The unperturbed part of the thermal state condition is given in the form

$$a|W(t_o)>> = f\tilde{a}^{\dagger}|(t_o)>>, \tag{4.9}$$

$$\tilde{a}|W(t_o)>> = \sigma f a^{\dagger}|W(t_o)>>, \tag{4.10}$$

where f is some c-number function. From equation (3.11) we see $f = f(0)$. By inspecting the symmetric property of a system, we can write down the general form of $\hat{H}$ in terms of the superoperators a, $a^{\dagger}$, $\tilde{a}$ and $\tilde{a}^{\dagger}$. Then the most basic structure of the Tildian Hamiltonian $\hat{H}$, which includes the dissipation effect of the system, is determined by the basic requirements A2~A6. This will be illustrated in section §5.

In the basic requirement A6, we can put in information about the symmetry of the stationary thermal vacuum ket-vector in the Schrodinger

representation $|W(\infty)\rangle\rangle$, which reveals itself in the structure of the real part of $i\hat{H}$. In other words, the symmetry breaking effect can be put into the structure of the real part of the $i\hat{H}$ through A6.

§5. A Phase-Invariant Bilinear Model

We now give an example of how axioms one through seven of section 4, are used in a calculation. Since the unperturbed thermal state condition equations (4.7) and (4.8) in A6 and equations (4.7) and (4.10) in A7 are invariant under the phase transformation $a \to a\exp(i\theta)$, $\hat{H}$ should assume this phase invariance. Thus, the general form of $\hat{H}$ is writen in the form

$$\hat{H} = h_1 a^\dagger a + h_2 \tilde{a}^\dagger \tilde{a} + h_3 \tilde{a} a + h_4 \tilde{a}^\dagger a^\dagger + h_o. \tag{5.1}$$

where $h=h'+ih''$ with h' and h'' real. The basic requirement A2 makes $\hat{H}$ Tildian. Then (5.1) reduces to

$$\hat{H} = h_1'(a^\dagger a - \tilde{a}^\dagger \tilde{a}) + ih_1''(a^\dagger a + \tilde{a}^\dagger \tilde{a}) + ih_3'' \tilde{a} a + ih_4'' \tilde{a}^\dagger a^\dagger + ih_o''. \tag{5.2}$$

The basic requirements A3 and A4 gives us relations between the h" terms as

$$h_o'' + \sigma h_4'' = 0, \tag{5.3a}$$

$$2h_1'' + h_3'' + h_4'' = 0. \tag{5.3b}$$

The basic requirements A5 and A6 gives us another relation between the h" terms as

$$h_o'' + \sigma h_3'' \bar{f}_\sigma = 0, \tag{5.4a}$$

$$2h_1'' + h_3'' \bar{f}_\sigma + h_4'' \bar{f}_\sigma^{-1} = 0, \tag{5.4b}$$

where

$$\bar{f}_\sigma = \sigma \bar{f}. \tag{5.5}$$

In deriving (5.4), we used the unperturbed thermal state condition (4.6) in A6 as it may be consistent with the phase-invariant bilinear model. From (5.4a), we see that $\bar{f}$ is a real quantity. Note that (5.3) and (5.4) are not independent. They reduce to

$$h_o'' = -\sigma h_4'', \tag{5.6a}$$

$$h_1'' = -\frac{1}{2}(h_3'' + h_4''), \tag{5.6b}$$

$$h_4'' = h_3''\bar{f}_\sigma. \tag{5.6c}$$

If we introduce real quantities ε, κ_1 and κ_2 by the definitions

$$\varepsilon = h_1', \tag{5.7a}$$

$$2\kappa_1 = h_3'', \tag{5.7b}$$

$$2\kappa_2 = h_4'', \tag{5.7c}$$

we finally obtain the general form of the Tildian Hamiltonian $\hat{H}$ for the phase- invariant bilinear model, which satisfies the basic requirement A2 ~ A6, as

$$\hat{H} = \varepsilon(a^\dagger a - \tilde{a}^\dagger\tilde{a}) - i(\kappa_1+\kappa_2)(a^\dagger a + \tilde{a}^\dagger\tilde{a}) + i2\kappa_1\tilde{a}a + i2\kappa_2\tilde{a}^\dagger a^\dagger - i2\sigma\kappa_2, \tag{5.8}$$

with the relation

$$\kappa_2 = \kappa_1\bar{f}_\sigma. \tag{5.9}$$

Using the Heisenberg equations of motion:

$$\partial_t a(t) = i[\hat{H}, a(t)]$$

$$= -i[\varepsilon - i(\kappa_1+\kappa_2)]a(t) + 2\sigma\kappa_2\tilde{a}^{\dagger\dagger}(t), \tag{5.10a}$$

$$\partial_t a^{\dagger\dagger}(t) = i[\hat{H}, a^{\dagger\dagger}(t)]$$

$$= i[\varepsilon - i(\kappa_1+\kappa_2)]a^{\dagger\dagger}(t) - 2\kappa_1\tilde{a}(t), \qquad (5.10b)$$

and the basic requirements A4, we obtain

$$\partial_t \langle\langle 1|a^{\dagger\dagger}(t)a(t) = -2(\kappa_1-\kappa_2)\langle\langle 1|a^{\dagger\dagger}(t)a(t) + 2\sigma\kappa_2. \qquad (5.11)$$

By applying the thermal vacuum ket-vector $|W(t_o)\rangle\rangle$ to (5.11), we have

$$\partial_t n(t-t_o) = -2(\kappa_1-\kappa_2)n(t-t_o) + 2\sigma\kappa_2, \qquad (5.12)$$

where we defined $n(t-t_o)$ by [c.f. (3.24)]

$$n(t-t_o) = \langle\langle 1|a^{\dagger\dagger}(t)a(t)|W(t_o)\rangle\rangle. \qquad (5.13)$$

In the limit $t\to\infty$, we obtain from (5.12)

$$n(\infty) = \langle\langle 1|a^{\dagger}a|W(\infty)\rangle\rangle = \frac{\sigma\kappa_2}{\kappa_1-\kappa_2}. \qquad (5.14)$$

Inspecting (5.11) and (5.14), we know that it will be convenient to introduce positive quantities κ and $\bar{n}$ by

$$\kappa = \kappa_1 - \kappa_2, \qquad (5.15a)$$

$$\bar{n} = \frac{\sigma\kappa_2}{\kappa_1-\kappa_2} = (\bar{f}^{-1}-\sigma), \qquad (5.15b)$$

where we used (5.9) in the second equality of (5.15b). The reason, that we can determine that κ and $\bar{n}$ should be positive is the stability of the system.

Then the Hamiltonian (5.8) reduces to

$$\hat{H} = \hat{H}_o + i\hat{\Pi} \qquad (5.16a)$$

with

$$\hat{H}_o = \varepsilon(a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a}) \qquad (5.16b)$$

$$\hat{\Pi} = -\kappa[(1+2\bar{n}_\sigma)(A^\dagger a + \tilde{a}^\dagger\tilde{a}) - 2(1+\bar{n}_\sigma)\tilde{a}a - 2\bar{n}_\sigma\tilde{a}^\dagger a^\dagger] - 2\sigma\kappa\bar{n}_\sigma, \tag{5.16c}$$

where

$$\bar{n}_\sigma = \sigma\bar{n}. \tag{5.17}$$

It should be noted that the Tildian Hamiltonian (5.16) has exactly the same form as the unperturbed part of the Tildian Hamiltonian[1] which was obtained by eliminating the reservoir variables. Note, however, that the entire consideration in this paper does not need any reference to the reservoir.

The two-point Green's function

$$G^{\alpha\beta}(t,s) = -i\langle\langle 1|T[a^\alpha(t)\bar{a}^\beta(s)]|W(t_o)\rangle\rangle, \tag{5.18}$$

is easily evaluated by rewriting it in terms of the quasi-particle superoperator defined by equation (3.15) and its tilde conjugate, and by using a Wick-type formula. The result has the same form as that given in ref. 1 for the semi-free field, and therefore, has the damping factor in time. When we introduce an interaction, we make use of $\hat{H}$ in this section as the unperturbed Tildian Hamiltonian. In this case the two-point Green's function in (5.18) becomes the internal lines in the Feynman diagrams. Note that in the expression of (5.18) we introduced the thermal doublet notation defined by

$$a^{\alpha}(t) = \begin{pmatrix} a(t) \\ \tilde{a}^{\dagger\dagger}(t) \end{pmatrix}, \quad \bar{a}^{\alpha}(t) = (a^{\dagger\dagger}(t), \tilde{a}(t)) I_{\sigma}, \tag{5.19}$$

where

$$I_{\sigma} = \begin{pmatrix} 1, & 0 \\ 0, & -\sigma \end{pmatrix}. \tag{5.20}$$

The time derivative of $\Omega(t)$ for the phase-invariant bilinear model is given by

$$d_t \Omega(t) = \langle\langle 1 | (\hat{\Pi} + \hat{\Pi}^{\dagger}) | W(t) \rangle\rangle, \tag{5.21}$$

with (5.16c). By using the thermal state condition properly (see Appendix A for details), we obtain

$$d_t \ell n \Omega(t) = \begin{cases} 4\kappa \dfrac{1}{1+2n(t-t_o)} [n(t-t_o)-\bar{n}], & \text{for boson} \\ 4\kappa \dfrac{1-2n(t-t_o)}{[1-n(t-t_o)]^2+n(t-t_o)^2} [n(t-t_o)-\bar{n}], & \text{for fermion.} \end{cases} \tag{5.22}$$

where $n(t-t_o)$ is defined by (5.13) the explicit form of which is given by

$$n(t) = \bar{n} + [n(o)-\bar{n}] e^{-2\kappa t}. \tag{5.23}$$

If f and $\bar{f}$ are equal to $\exp(-\beta_o \epsilon)$ and $\exp(-\beta\epsilon)$, respectively, with $T_o = 1/\beta_o$ and $T = 1/\beta$ ($k_B = 1$) being the initial and final temperatures of the system, respectively, we obtain

$$\bar{n} = (e^{\beta\epsilon} - \sigma)^{-1}, \tag{5.24a}$$

$$n(o) = (e^{\beta_o \epsilon} - \sigma)^{-1}, \tag{5.24b}$$

and

$$0 < n(t-t_o) < \infty, \quad \text{for boson,} \tag{5.25a}$$

$$0 < n(t-t_o) < \frac{1}{2}, \quad \text{for fermion,} \tag{5.25b}$$

as can be seen from (5.23) and (5.24). Then (5.22) tells us that

$$d_t S(t) > 0, \quad \text{for } T > T_o, \tag{5.26a}$$

$$d_t S(t) < 0, \quad \text{for } T < T_o. \tag{5.26b}$$

From (5.23) and (5.26), we can see that the phase-invariant bilinear model with this particular choice for f and $\bar{f}$ describes an open system coupling to the particle reservoir with temperature T if the chemical potential of the reservoir remains constant.

§6. Generating Functional of Non-equilibrium TFD (12)

Up to this point we have, in part III, introduced superoperators and supervectors. We have developed the different representations for the equation of motion for our thermal state vectors and have shown what the form of the thermal state condition is and how it is used to produce the quasi-particle operators. With quasi-particle operators, most all of the many quantum field theoretical techniques are at our disposal. We then listed the seven axioms which made this whole development possible and showed how they worked with the example of a bilinear model. We now finish this part with the presentation of the generating functional for non-equilibrium TFD, and concluding comments.

To obtain a generating functional, we introduce external fields $K^*(t)$, $K(t)$, $\tilde{K}^*(t)$ and $\tilde{K}(t)$ which are conjugate to the superoperators a, $a^\dagger$, $\tilde{a}$ and $\tilde{a}^\dagger$, respectively. Note that we are going to consider independent variations of four external fields. Then the "master equation" of the system under the influence of the external fields becomes

$$\partial_t |W(t)>> = -i\hat{H}_t |W(t)>>, \tag{6.1}$$

with

$$\hat{H}_t = \hat{H}' + \hat{H}_{I,t}, \tag{6.2}$$

where

$$\hat{H}' = \hat{H}_o + i\hat{\Pi} \tag{6.3}$$

$$\hat{H}_{I,t} = \sum_k [\bar{K}_k^\alpha(t)a_k^\alpha + \bar{a}_k^\alpha K_k^\alpha(t)]. \tag{6.4}$$

where $\hat{H}_o$ and $\hat{\Pi}$ were obtained for a bilinear model in section 5, and can similarly be obtained for any other model. We have introduced the thermal doublet notation

$$K_k^\alpha(t) = \begin{pmatrix} K_k(t) \\ \tilde{K}^*_k(t) \end{pmatrix}, \quad \bar{K}_k^\alpha(t) = (K_k^*(t), \tilde{K}_k(t))I_\sigma, \tag{6.5}$$

whose elements satisfy the double-tilde conjugation rule

$$\tilde{\tilde{K}}_k(t) = \sigma K_k(t), \quad \tilde{\tilde{K}}_k^*(t) = \sigma K_k^*(t), \tag{6.6}$$

and the σ-commutativity

$$[K_k^\alpha(t), \bar{K}_\ell^\beta(t)]_\sigma = 0, \quad [K_k^\alpha(t), \bar{a}_\ell^\beta(t)]_\sigma = 0, \text{ etc..} \tag{6.7}$$

and where

$$I_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -\sigma \end{pmatrix}. \tag{6.7a}$$

In the interaction representation with the interaction "Hamiltonian" $\hat{H}_{I,t}$, the "master equation" (6.1) becomes

$$\partial_t |W(t)\rangle\rangle_I = -i\hat{H}_I(t)|W(t)\rangle\rangle_I, \tag{6.8}$$

where

$$|W(t)\rangle\rangle_I = \hat{S}_o^{-1}(t-t_o)|W(t)\rangle\rangle, \tag{6.9}$$

$$\hat{H}_I(t) = \hat{S}_o^{-1}(t-t_o)\hat{H}_{I,t}\hat{S}_o(t-t_o)$$

$$= \sum_k [\bar{K}_k^\alpha(t) a_k^\alpha(t) + \bar{a}_k^\alpha(t) K_k^\alpha(t)], \qquad (6.10)$$

with

$$\hat{S}_o(t) = \exp[-i\hat{H}_o t]. \qquad (6.11)$$

We have introduced the thermal doublet

$$a_k^\alpha(t) = \begin{pmatrix} a_k(t) \\ \tilde{a}_k^{\dagger\dagger}(t) \end{pmatrix}, \quad \bar{a}_k^\alpha(t) = (a_k^{\dagger\dagger}(t), \tilde{a}_k(t))I_\sigma, \qquad (6.12)$$

whose elements are defined by equations (3.9) and (3.10).

Using the time evolution operator of the thermal vacuum ket-vector state $|W(t)>>_I$:

$$\hat{U}(t,t_o) = T \exp[-i \int_{t_o}^{t} ds\, \hat{H}_I(s)], \qquad (6.13)$$

which satisfies the equation of motion

$$\partial_t \hat{U}(t,t_o) = -i\hat{H}_I(t)\hat{U}(t,t_o), \qquad (6.14)$$

which together with (3.8) leads to

$$|W(t)>>_I = \hat{U}(t,t_o)|W(t_o)>>_I = \hat{U}(t,t_o)|W(t_o)>>. \qquad (6.15)$$

Following the theory of generating functional in the ordinary quantum field theory[13], we introduce the thermal generating functional:

$$Z[K,\tilde{K}] = <<1|\hat{U}(\bar{t},t_o)|W_S(t_o)>>. \qquad (6.16)$$

The generating functional (6.16) can be written in terms of the quasi-particle operator. In the following, we use the notation $Z_\gamma[K_\gamma,\tilde{K}_\gamma]$ for (6.16) in which $\hat{H}_I(t)$ in

$\hat{U}(\bar{t},t_o)$ is written in terms of the quasi-particle operator as

$$\hat{H}_I(t) = \sum_k [\bar{K}^{\alpha}_{\gamma,k}(t)\gamma^{\alpha}_k(t) + \bar{\gamma}^{\alpha}_k(t)K^{\alpha}_{\gamma,k}(t)], \qquad (6.17)$$

where we have introduced the thermal doublet

$$\gamma^{\alpha}_k(t) = \begin{pmatrix} \gamma_k(t) \\ \tilde{\gamma}^{\dagger}_k(k) \end{pmatrix}, \quad \bar{\gamma}^{\alpha}_k(t) = (\gamma^{\dagger}_k(t), \tilde{\gamma}_k(t))I_{\sigma}, \qquad (6.18)$$

whose elements are defined by equations (6.15) and (6.16). They are written in the thermal doublet notation as

$$\gamma^{\alpha}_k(t) = B_k(t-t_o)^{\alpha\beta}a^{\beta}_k(t), \quad \bar{\gamma}^{\alpha}_k(t) = \bar{a}^{\beta}_k(t)B^{-1}_k(t-t_o)^{\beta\alpha} \qquad (6.19)$$

where

$$B_k(t)^{\alpha\beta} = Z^{1/2}_k(t) \begin{pmatrix} 1 & -f_k(t) \\ -\sigma & 1 \end{pmatrix}. \qquad (6.20)$$

We have

$$B^{-1}_k(t) = \tau_3 B_k(t)\tau_3, \qquad (6.21)$$

where

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (6.21a)$$

The external fields which are conjugate to the quasi-particle operators are given by

$$K^{\alpha}_{\gamma,k}(t) = B_k(t-t_o)^{\alpha\beta}K^{\beta}_k(t), \quad \bar{K}^{\alpha}_{\gamma,k}(t) = \bar{K}^{\beta}_k(t)B^{-1}_k(t-t_o)^{\beta\alpha} \qquad (6.22)$$

and they are σ-commutable.

A variation of the external fields induces a change in $\hat{H}_I$. Denoting this change by $\delta\hat{H}$, we obtain

$$\delta\hat{U}(\bar{t},t_o) = -i\int_{t_o}^{\bar{t}} dt\ \hat{U}(\bar{t},t_o)\delta\hat{H}(t)\hat{U}(t,t_o)$$

$$= -i\int_{t_o}^{\bar{t}} dt \sum_k \hat{U}(\bar{t},t)[\delta\bar{K}^{\alpha}_{\gamma,k}(t)\gamma^{\alpha}_k(t) + \bar{\gamma}^{\alpha}_k(t)\delta K^{\alpha}_{\gamma,k}(t)]\hat{U}(t,t_o).$$

This leads to

$$\delta \ell n Z_\gamma[K_\gamma,\tilde{K}_\gamma] = \delta Z_\gamma[K_\gamma,\tilde{K}_\gamma]/Z_\gamma,[K_\gamma,\tilde{K}_\gamma]$$

$$= -i\int_{t_o}^{\bar{t}} dt \sum_k [\delta\bar{K}^{\alpha}_{\gamma,k}(t)\langle\gamma^{\alpha}_k(t)\rangle + \langle\bar{\gamma}^{\alpha}_k(t)\rangle\delta K^{\alpha}_{\gamma,k}(t)], \quad (6.23)$$

where

$$\langle\gamma^{\alpha}_k(t)\rangle \equiv i\,\frac{\delta \ell n Z_\gamma[K_\gamma,\tilde{K}_\gamma]}{\delta\bar{K}^{\alpha}_{\gamma,k}(t)}$$

$$= \int_{t_o}^{\bar{t}} dt\ \langle\langle 1|U(\bar{t},t)\gamma^{\gamma}_k(t)U(t,t_o)|W_S(t_o)\rangle\rangle/Z_\gamma[K_\gamma,\tilde{K}_\gamma], \quad (6.24a)$$

$$\langle\bar{\gamma}^{\alpha}_k(t)\rangle \equiv i\,\frac{\delta \ell n Z_\gamma[K_\gamma,\tilde{K}_\gamma]}{\delta K^{\alpha}_{\gamma,k}(t)}$$

$$= \int_{t_o}^{\bar{t}} dt\ \langle\langle 1|\hat{U}(\bar{t},t)\bar{\gamma}^{\alpha}_k(t)\hat{U}(t,t_o)|W_S(t_o)\rangle\rangle/Z_\gamma[K_\gamma,\tilde{K}_\gamma]. \quad (6.24b)$$

By the thermal state condition, equations (3.17) and (3.18), $\langle\gamma^{\alpha}_k(t)\rangle$ and $\langle\bar{\gamma}^{\alpha}_k(t)\rangle$ satisfy the boundary conditions

$$\langle\gamma^{1}_k(t_o)\rangle = 0, \quad \langle\gamma^{2}_k(\bar{t})\rangle = 0, \quad (6.25a)$$

$$\langle\gamma^{-1}_k(\bar{t}) = 0, \quad \langle\bar{\gamma}^{2}_k(t_o)\rangle = 0. \quad (6.25b)$$

The quantity $\langle\gamma^{\alpha}_k(t)\rangle$ satisfies the equation of motion

$$\partial_t\langle\gamma^{\alpha}_k(t)\rangle = [-i\varepsilon_k\delta^{\alpha\beta} - \kappa_k\,\frac{1+\bar{n}_{\sigma k}}{1+\bar{n}_{\sigma k}(t)}\,\tau_3^{\alpha\beta}]\langle\gamma^{\beta}_k(t)\rangle - iK^{\alpha}_{\gamma,k}(t), \quad (6.26)$$

which is obtained from (6.24) together with (6.14), (6.14),

(6.19) and the Heisenberg equation of motion for $a_k^\alpha(t)$. With the boundary condition (6.25a), the equation of motion (6.26) is solved as

$$\langle \gamma_k^\alpha(t)\rangle = \int_{t_o}^{\bar{t}} ds\; g_k^{\alpha\beta}(t,s) K_{\gamma,k}^\beta(s), \tag{6.27}$$

where $g_k^{\alpha\beta}(t,s)$ is the two-point Green's function of the quasi-particle operator defined by

$$g_k^{\alpha\beta}(t,s) = -i\langle\langle 1|T[\gamma_k^\alpha(t)\bar{\gamma}_k^\beta(s)]|W_S(t_o)\rangle\rangle$$

$$= \begin{pmatrix} Z_k^{1/2}(s-t_o)Z_k^{-1/2}(t-t_o)G_k^r(t-), & 0 \\ 0 & , Z_k^{1/2}(t-t_o)Z_k^{-1/2}(s-t_o)G_k^a(t-s) \end{pmatrix} \tag{6.28}$$

with

$$G_k^r(t) = -i\theta(t)\exp[-i(\varepsilon_k - i\kappa_k)t], \tag{6.29a}$$

$$G_k^a(t) = i\theta(-t)\exp[-i(\varepsilon_k + i\kappa_k)t]. \tag{6.29b}$$

The tilde conjugate of the elements (6.27) gives us the result

$$\langle \bar{\gamma}_k^\alpha(t)\rangle = \int_{t_o}^{\bar{t}} ds\; \bar{K}_k^\beta(s) g_k^{\beta\alpha}(s,t). \tag{6.30}$$

To obtain (6.30), we used the relations

$$\tilde{G}_k^r(t) = G_k^a(-t), \quad \tilde{G}_k^a(t) = G_k^r(-t). \tag{6.31}$$

By substituting (6.27) and (6.30) into (6.23), we finally obtain the expression

$$Z_\gamma[K_\gamma, \tilde{K}_\gamma] = \exp[-i \int_{t_o}^{\bar{t}} dt \int_{t_o}^{\bar{t}} ds \sum_k \bar{K}_{\gamma,k}^\alpha(t) g_k^{\alpha\beta}(t,s) K_{\gamma,k}^\beta(s)],$$

(6.32)

which can be written in terms of the original external forces as

$$Z[K,\tilde{K}] = \exp[-i \int_{t_o}^{\bar{t}} dt \int_{t_o}^{\bar{t}} ds \sum_k \bar{K}_k^{\alpha}(t) G_k^{\alpha\beta}(t,s) K_k^{\beta}(s)], \quad (6.33)$$

where

$$G_k^{\alpha\beta}(t,s) = [B_k^{-1}(t-t_o) g_k(t,s) \beta_k)(s-t_o)]^{\alpha\beta}$$
$$= [I_{\bar{\sigma}} W_k(t-t_o) g_k(t,s) W_k^{-1}(s-t_o) I_{\bar{\sigma}}]^{\alpha\beta}, \quad (6.34)$$

with

$$W_k(t) = I_\sigma B_k(t) I_\sigma. \quad (6.35)$$

Recall that in the special case where the initial state is of the grand canonical distribution with temperature $T_o = \beta_o^{-1}$ we had equations (5.24a) and (5.24b). For this choice of the initial state, the generating functional (6.33)with σ=1 (ie., for boson) reduces to that derived by Schwinger[14] (see also ref. 15). Note that, in this case, the conditions in (6.25), which followed from the thermal state condition and which played an essential role in derivation of (6.33), are equivalent to Schwinger's condition at both edges of the closed path. This clarifies the relation between the boundary conditions in Schwinger's formalism and the thermal state condition in TFD.

Finally we introduce an interaction, $\hat{H}_I$, in the

system, which induces the dynamical correlations. Then, the generating functional becomes

$$Z[K,\tilde{K}] = \langle\langle 1|\hat{U}'(\bar{t},t_o)|W_S(t_o)\rangle\rangle, \tag{6.36}$$

with

$$\hat{U}'(t,t_o) = T\exp[-i\int_{t_o}^{t} ds\ \hat{H}_I'(s)], \tag{6.37}$$

where

$$\hat{H}_I'(t) = \hat{S}_1^{-1}(t,t_o)\hat{H}_I(t)\hat{S}_1(t,t_o). \tag{6.38}$$

Here $\hat{S}_1(t,t_o)$ is defined by

$$\partial_t\hat{S}_1(t,t_o) = -i\hat{H}_I(t)\hat{S}_1(t,t_o), \quad \hat{S}(t_o,t_o) = 1, \tag{6.39}$$

with

$$\begin{aligned}\hat{H}_I(t) &= \hat{S}_o(t-t_o)\hat{H}_I\hat{S}_o^{-1}(t-t_o)\\ &= \hat{H}_I[a_k^{\alpha}(t),\bar{a}_k^{\beta}(t)].\end{aligned} \tag{6.40}$$

Since $\hat{H}_I(t)$ in (6.38) never acts as an external vertex, we can write

$$Z'[K,\tilde{K}] = \exp\{-i\int_{t_o}^{\bar{t}} dt\ \hat{H}_I'[\frac{1}{i}\frac{\delta}{\delta\bar{K}_k^{\alpha}(t)}, \frac{1}{i}\frac{\delta}{\delta K_k^{\beta}(t)}]\}Z[K,\tilde{K}]. \tag{6.41}$$

Since the fields $a_k^{\alpha}(t)$ and $\bar{a}_k^{\alpha}(t)$ satisfy the canonical commutators, $\hat{H}_I'$ in (6.41) can be replaced by the corresponding interaction Lagrangian $\hat{L}_I$.

The expression of the generating functional for the time-ordered many-point Green's functions of nonequilibrium TFD provide us with the Feynman-like diagram rules in nonequilibrium TFD which are just those expected in the canonical approach to nonequilibrium TFD.[1,16]

Note that the direction, from which the functional derivatives operate on $Z[k,\tilde{K}]$ in (6.41), should be properly taken because of (6.7).

This then ends our discussion of TFD for non-equilibrium situations. Its development has been carrying on at a tremendous pace. The generalized form of the type-one and type-two field equations, and the free fields which satisfy them have now been formulated and hence it seems we have a well defined dissipative internal line for Feynman diagrams. With this realization, work has begun in the description of a laser system using TFD.

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Part IV. Comparisons Between TFD and Other Thermal Theories

§1. TFD and Path Ordering Method

Matsubara's Green's function method[1] is commonly used for finite temperature calculations, especially when only static quantities are being considered. Since Matsubara's method can accommodate only one dynamical parameter, namely imaginary time or temperature, calculation of dynamical quantities often requires a process of analytical continuation.[2,3] Real-time extensions of the perturbational rules in Matsubara's method go under the name of the complex time "path ordering method[4,5,6]. It is the purpose of this section to show that TFD and the path ordering method yield the same perturbation expansion for two point Green's functions. Our comparison can not be carried out at a deeper level because the path ordering method is not a quantum field theoretic formalism but only a method for establishing Green's functions. This section will deal with equilibrium situations as all of the above referenced thermal theories are suited only to handle such cases.

Consider a complex scalar field $\psi(\vec{x},z)$, where z is the complex time variable, the dynamics of ψ along a trajectory on the z plane are determined by the Hamiltonian H. Hence we write

$$\psi(z) = \exp(izH)\psi(o)\exp(-izH) \qquad (1.1)$$

We take our trajectory as path 'C'; starting at point 'τ' and ending at point $\tau - i\beta$, in the z-plane. The path 'C' is so chosen to pass through points, $z_1, z_2 \ldots z_n$ so that we may

define a statistical average as

$$\langle\psi(z_1)\ldots\psi(z_n)\rangle = \frac{\mathrm{tr}[e^{-\beta H}T_c(\psi(z_1)\ldots\psi(z_n)]}{\mathrm{tr}[e^{-\beta H}]} \quad . \quad (1.2)$$

'T_c' is the path ordering operator which is equal to the time ordering operator 'T' when Re|z| is increasing and is equal to the anti-time ordering operator '$\tilde{T}$' when Re|z| is decreasing, as we proceed from τ to $\tau - i\beta$. In the interaction representation with

$$\psi_o(z) = e^{izH_o}\psi(o)e^{-izH_o} \quad (1.3)$$

and

$$u(\tau - i\beta,\tau) = \exp\Big(-i\int_{\tau}^{\tau-i\beta}_{\text{along c}} dz\ H_I(z)\Big) \quad (1.4)$$

we can express equation (1.2) as[4,5,6]

$$\langle\psi(z_1)\ldots\psi(z_n)\rangle = \frac{\langle T_c\ u(\tau-i\beta,\tau)\psi_o(z_1)\ldots\psi_o(z_n)\rangle_o}{\langle T_c\ u(\tau-i\beta,\tau)\rangle_o} \quad . \quad (1.5)$$

The symbol $\langle A\rangle_o$ is the same as that of $\langle A\rangle$ defined in equation (1.2) except H is replaced by H_o. In equation (1.4) we used

$$H_I(z) = e^{iH_oz}H_I\,e^{-iH_oz} \quad . \quad (1.6)$$

We now consider the case where $z_1\ldots z_n$ all lie on the real time axis (ie. $(z_1\ldots z_n) \to (t_1\ldots t_n)$), and we take $\tau \to -\infty$ with path 'C' covering the whole real time axis ($t \to \infty$) and returning to '$-\infty$' with a constant imaginary value of $-i\sigma$. In this limit the effect of the interaction Hamiltonian on the two portions $(+\infty, +\infty - i\sigma)$ and $(-\infty - i\sigma, -\infty - i\beta)$ may be

considered disconnected from the fields $\psi_o(t_1)...\psi_o(t_n)$ and hence we obtain

$$\langle\psi_o(t_1)...\psi_o(t_n)\rangle = \frac{\langle T_c\ u\ \psi_o(t_1)...\psi_o(t_n)\rangle}{\langle T_c\ u\rangle_o} . \quad (1.7)$$

where

$$u = \exp\left(-i\int_{-\infty}^{\infty} dt(H_I(t) - H_I(t - i\sigma))\right). \quad (1.8)$$

The free Hamiltonian has the form

$$H_o = \int d^3x\ \psi_o^\dagger(x)\omega(\nabla)\psi_o(x). \quad (1.9)$$

Expanding $\psi_o(x)$ as in equation (I.8.19) we have

$$\psi_o(x) = (2\pi)^{-3/2}\int d^3k\ a(\vec{k})\exp(i\vec{k}.\vec{x} - i\omega(\vec{k})t) \quad (1.10)$$

and the four two-point functions which are now relevant (ie. propagation from the $t \to t'$, $t \to t' - i\sigma$, $t - i\sigma \to t'$, and $t - i\sigma \to t' - i\sigma$) can be calculated and the results[6] may be written in a two dimensional matrix equation as

$$\langle T_c \begin{matrix}\psi_o(\vec{x},t)\\ \psi(x,t-i\sigma)\end{matrix} (\psi^\dagger(\vec{x}',t'),\psi^\dagger(\vec{x},t'-i\sigma))\rangle_o =$$

$$\frac{i}{(2\pi)^4}\int d^4k\ e^{-ik(x-x')}\left\{\frac{1}{k_o-\omega(k)+i\delta}\ \frac{1}{e^{\beta\omega(k)}-1}\times\right.$$

$$\begin{pmatrix} e^{\beta\omega(k)} & e^{(\beta/2+\gamma)\omega(k)} \\ e^{(\beta/2-\gamma)\omega(k)} & 1 \end{pmatrix} - \frac{1}{k_o-\omega(k)-i\delta}\ \frac{1}{e^{\beta\omega(k)}-1}\times$$

$$\left.\begin{pmatrix} 1 & e^{(\beta/2+\gamma)\omega(k)} \\ e^{(\beta/2-\gamma)\omega(k)} & e^{\beta\omega(k)} \end{pmatrix}\right\} \quad (1.11)$$

where $\gamma = \sigma - \beta/2$ ($\gamma \in [-\beta/2, \beta/2]$). If we choose $\gamma = 0$ we find that equation (1.11) is precisely the perturbative expansion for a complex scalar field derived in section 6 of part II, namely equation (6.24). It can be shown that this correspondence between the path ordering method and TFD holds generally[7] for other moments and fields.

The path ordering method seems to be expressing the need for the doubling of the degrees of freedom, in that it separates the fields on the forward and reverse paths with an imaginary time $-i\sigma$. Of course $\psi_o(t)$ and $\psi_o(t - i\sigma)$ do not commute and hence no such doubling of the degrees of freedom is contained in the path ordering method. An important point is that there is no systematic way of stipulating how the fields on the two paths are correlated, eg. we can not put in a model's irreversible nature. Hence there is no way to develop a non-equilibrium theory around the path ordering method except the more general Schwinger formalism.

Schwinger[8] developed a non-equilibrium path ordering method under three assumptions. First, the temperature of the relevant system is changed through its coupling to a reservoir. Secondly, the reservoir is at equilibrium with itself at all times, and lastly that the full density matrix is initially separable ($\rho(t_o)_{\text{full}} = \rho(t_o)_{\text{system}} \times \rho(t_o)_{\text{reservoir}}$). By introducing evolutionary

direction sensitive external driving forces, Schwinger cast his path ordering method into a functional generating form. Variations of these forces will generate the moments, as was done in part III. The first objective of Schwinger's was to eliminate (project out) the reservoirs degrees of freedom. This was done by taking the partial trace average over just the reservoirs degrees of freedom. This was easily done using the path ordering method described above because the reservoir is at equilibrium. In order to accomplish the integration along path 'C' one needs boundary conditions. Denoting by $U_+(t_o,t_1)$ $[U_-(t_o,t_1)]$ the time evolution operator on the real time path increasing [decreasing] part of path 'C' we have that the expectation of reservoir variable $a(t_o)$ is given by

$$\langle a(t_o)\rangle = \mathrm{Tr}(U_-(t_o,t_1)U_+(t_o,t_1)a(t_o)\rho(t_o)) \quad (1.12)$$

at the initial point of path C, and

$$\langle a(t_o-i\varepsilon)\rangle = \mathrm{Tr}\big(a(t - i\varepsilon)U_-(t_o,t_1)\rho(t_o)\big). \quad (1.13)$$

at the final point of the return part of path C.
Where we have path ordered the operators and hence the path ordering operator 'T_c' does not appear. Due to the cyclic invariance of the trace and the separability of $\rho(t_o)$, we have

$$\langle a(t_o - i\varepsilon)\rangle = \mathrm{Tr}\big(U_-(t_o,t_1)U_+(t_o,t_1)\rho(t_o)a(t - i\varepsilon)\big) \quad (1.14)$$

$$= \mathrm{Tr}\Big(U_-(t_o,t_1)U_+(t_o,t_1)\frac{e^{-\beta\omega a^\dagger a}}{1 + e^{-\beta\omega}}\rho'(t_o)_{\text{system}}\, a(t - i\varepsilon)\Big) \quad (1.15)$$

$$= \mathrm{Tr}\left(U_-(t_o,t_1)U_+(t_o,t_1)e^{-\beta\omega a^\dagger a}a(t - i\varepsilon)\frac{\rho'(t_o)}{1 + e^{\beta\omega}}\right) \quad (1.16)$$

$$= \mathrm{Tr}\left(U_-(t_o,t_1)U_+(t_o,t_1)e^{-\beta\omega a^\dagger a}a(t - i\varepsilon)e^{\beta\omega a^\dagger a}\rho(t_o)\right) \quad (1.17)$$

$$= \langle a(t_o)e^{\beta\omega}\rangle. \quad (1.18)$$

Hence Schwinger used equation (1.18) as the boundary condition. The effective Hamiltonian then contains information about the final (reservoir) temperature of the system. Due to the fact that while using functional methods one works in an interaction picture with respect to the external forces, this final-temperature dependence of the Hamiltonian manifests itself in the (thermal) time dependent annihilation and creation operators. When we calculate the Green's functions along the path 'C', which depicts the complete evolution of the system, we use the same boundary conditions as we did for the reservoir average, but with $\beta = \beta_o$, which is the initial temperature of the relevant system. One notices that the temperature and time averaging are treated as very separate matters in Schwinger's formalism and that an explicit form for the coupling between reservoir and system must be assumed.

Equation (1.18) is the thermal state condition for the initial equilibrium situation. We showed explicitly in part III how one describes an equilibrium system evolving into another equilibrium situation at a different temperature. To zeroth order we have shown (part III) that replacing $e^{-\beta_o\omega(k)}$

in equation (1.18) with a more general c-number function f_k leads to an unperturbed non-equilibrium formalism for which Schwinger's formalism would be able to generate Green's functions. Of course no higher levels of "thermal perturbation" can be developed in Schwinger's formalism, nor is Schwinger's formalism able to use any of the calculational techniques of quantum field theory because it is not a quantum field theory, for it can not specify the quasi-particle operators.

§2. TFD and Sub-Dynamics

A way of possibly introducing irreversibility, was introduced by Prigogine[9,10,11]; it is the so called sub-dynamics. An argument given in sub-dynamics can be summarized as follows.[12]

The density operator satisfies the Liouville equation and hence can be written using superoperator 'L' defined by

$$LX \equiv [H,X]_- \tag{2.1a}$$

$$L^\dagger X = [H^\dagger,X]_- = LX \tag{2.1b}$$

as

$$\frac{\partial}{\partial t}\rho(t) = -iL\rho(t), \tag{2.2}$$

which is reversible.

We move into the so called causal "representation" through the action of superoperator $\Lambda(L)$ as

$$\rho(t)_{causal} = \tilde{\rho}(t) = \Lambda^{-1}(L)\rho(t). \tag{2.3}$$

where "~" is not the tilde operation of TFD. Hence we have

$$\frac{\partial}{\partial t}\tilde{\rho}(t) = i\,\Lambda^{-1}(L)L\,\Lambda(L)\tilde{\rho}(t) \tag{2.4}$$

$$\equiv -i\,\Phi(L)\tilde{\rho}(t). \tag{2.5}$$

We define the expectation value of observable A as

$$\langle A\rangle \equiv tr(A^\dagger\rho(t)) \equiv tr(\tilde{A}^\dagger\tilde{\rho}(t))$$

$$\equiv (A,\rho(t)) \tag{2.6}$$

Since equation (2.2) can be formally solved as

$$\rho(t) = e^{-iLt}\rho(o) \tag{2.7a}$$

$$\equiv u_L(t)\rho(o) \tag{2.7b}$$

we have

$$\tilde{\rho}(t) = \Lambda^{-1}(L)\, e^{-iLt}\Lambda(L)\tilde{\rho}(o) \tag{2.8}$$

$$\equiv \tilde{u}_L(t)\tilde{\rho}(o). \tag{2.9}$$

Using the relation[9]

$$(A,QB) = (Q^{\dagger}A,B), \tag{2.10}$$

we may write equation (2.6) as

$$\langle A\rangle = (A(o),e^{-iLt}\rho(o)) \tag{2.11}$$

$$= (e^{+iL^{\dagger}t}\bar{A}(o),\rho(o)) \tag{2.12}$$

$$= (e^{iLt}A(o),\rho(o)) \tag{2.13}$$

$$= (A(t),\rho(o)). \tag{2.14}$$

Equation (2.11) gives the Schrodinger picture expectation value of A and yields

$$\langle A\rangle = (A,\Lambda(L)\,\tilde{u}_L(t)\Lambda^{-1}(L)\rho(o)) \tag{2.15}$$

$$= (\Lambda^{\dagger}(L)A,\ \tilde{u}_L(t)\tilde{\rho}(o)) \tag{2.16}$$

$$= (\Lambda^{\dagger}(L)A,\tilde{\rho}(t)). \tag{2.17}$$

With equation (2.6) we define $\tilde{A}$ such that

$$(A,\rho(t) = (\tilde{A},\tilde{\rho}(t)) \tag{2.18}$$

and hence

$$\Lambda^{\dagger}(L)A = \tilde{A}. \tag{2.19}$$

So equation (2.18) may be written

$$\langle A\rangle = (\tilde{u}_L^{\dagger}(t)\tilde{A},\tilde{\rho}). \tag{2.20}$$

Similarly, equation (2.13) gives the Heisenberg picture's expectation value of A and can be written

$$\langle A\rangle = (\tilde{u}_{-L}(t)\tilde{A},\tilde{\rho}). \tag{2.21}$$

Since the two pictures should give the same expectation value, we have that

$$\tilde{u}_L^{\dagger}(t) = \tilde{u}_L^{-1}(t), \tag{2.22}$$

$\tilde{u}_L(t)$ is said to be star-Harmitian. In other words B(L) is star-Hermitian if

$$B(L)^{*} = B^{\dagger}(-L) = B^{-1}(L). \tag{2.23}$$

Equation (2.22) implies that

$$\Lambda^{*}(L) = \Lambda^{\dagger}(-L) = \Lambda^{-1}(L) \tag{2.24}$$

or $\Lambda(L)$ is also star-Hermitian. With equation (2.24) it can be shown that

$$(i\Phi(L))^{*} = i\Phi(L), \tag{2.25}$$

and since $\Phi(L)$ is the time translation generator in the causal picture we see a great deal of similarity between equation (2.25) and the Tildian property of $\hat{H}$. Defining

$$\Omega(t) = tr(\tilde{\rho}^{\dagger}(t)\tilde{\rho}(t)) \tag{2.26}$$

as in equation (3.19) of part III, we find that

$$-\frac{\partial}{\partial t}\Omega(t) = -tr(\tilde{\rho}^{\dagger}(t)(i\Phi^{\dagger} - i\Phi)\tilde{\rho}(t)), \tag{2.27}$$

which can be shown to be greater or equal to zero, for a closed system and hence with equation (3.18) of part III we have that

$$\frac{ds}{dt} \geq 0. \tag{2.28}$$

This is an elegant formalism and it all hinges on the causal transformation $\Lambda(L)$, which is determined through various very tedious means[9] and for the case of the Freidrichs model can be determined exactly. A problem exists in sub-dynamics

however, since $\Lambda(L)$ depends only on L it must commute with L and hence equations (2.4) and (2.5) give that

$$\Phi(L) = L. \qquad (2.29)$$

Hence no transformation has really been accomplished to date in the literature. Equations (2.26), (2.27) and (2.28) all reduce to zero. The similarities between equation (3.20) of part III

$$d_t\Omega(t) = -\langle\langle W^\dagger(t)|[(i\hat{H})^\dagger - (i\hat{H})]|W(t)\rangle\rangle \qquad (III.3.20)$$

and equation (2.27) above are striking and hence sub-dynamics does have something very important to offer. To make the causal transformation meaningful we must introduce, by hand, some external force which does not commute with L and hence will make what follows from equation (2.4) meaningful. This sounds much like what Schwinger did but sub-dynamics will work with this put-in-by-hand irreversibility in a much different way. The task of developing a meaningful $\Lambda(L)$ is similar to setting the thermal state condition so as to reflect the correlations of the system under study. However the latter is easily done in a very systematic way and is part of a whole quantum field theory. Our comparison between sub-dynamics and TFD then rests on the similarity between the forms of the various equation which make up sub-dynamics and TFD. Work towards a meaningful and model sensitive $\Lambda(L)$ would be interesting.

§3. TFD and c^*-algebra

Much work has been done[13] towards obtaining a general characterization of thermodynamical equilibrium states without reference to a particular state or representation of a system. We now know that such a characterization can be understood as the so called c^*-algebra of observables of the infinite (in the thermodynamical limit) system. C^*-algebra is an abstraction of the structure of bounded operators acting on a Hilbert space. We must of course relax this boundedness condition for our operator space. Some progress towards a formal relaxation of this condition can be found in reference 14.

C^*-algebra ($\mathcal{A}$) is a normed vector space with addition, multiplication and *-conjugation operations defined over it. Hence for A, B $\varepsilon \mathcal{A}$, and λ a c-number, we have

$$A^*,\ A+B,\ AB,\ \lambda A\ \varepsilon\ \mathcal{A} \tag{3.1}$$

$$(A+B)^* = A^* + B^* \tag{3.2}$$

$$(AB)^* = B^* A^* \tag{3.3}$$

$$(\lambda A)^* = \lambda^* A^* \tag{3.4}$$

$$(A^*)^* = A \tag{3.5}$$

$$||A||^2 = ||A^* A||. \tag{3.6}$$

A Gibbs state is given by the expectation functional $\omega(A)$ of operator set $\{A\}\varepsilon \mathcal{A}$,

$$\omega(A) \equiv \frac{tr(\exp(-\beta H)A)}{tr(\exp(-\beta H))} = \langle A \rangle \tag{3.7}$$

As an axiom, one finds[14] that for any two operators A(t) and B(t) in '$\mathcal{A}$' the KMS condition (section 8, part II) must be satisfied, ie.

$$F_{AB}(t) = \omega(A(t)B(t')) \; + \; F_{AB}(t + i\beta) = \omega(B(t')A(t)). \quad (3.8)$$

When a representation $R(\mathcal{A})$ is generated on the cyclic vector Ω,

$$\omega(A) = (\Omega, R(A)\Omega) \quad (3.9)$$

$$\mathcal{H}_\omega = \overline{R(\;)\Omega}, \quad (3.10)$$

it can be shown[14] that the commutant $\tilde{R}(\mathcal{A})$ of $R(\mathcal{A})$ must exist. By commutant we mean for $\tilde{R}(A) \in \tilde{R}(\mathcal{A})$ and $R(B) \in R(\mathcal{A})$ we have that

$$[\tilde{R}(A), R(B)]_- = 0. \quad (3.11)$$

Also the modular operator Δ and modular conjugation operator J necessarily exist such that

$$\langle H\varphi | J\psi \rangle = \langle \varphi | \psi \rangle^* \quad (3.12)$$

$$J^2 = 1 \quad (3.13)$$

$$J\, R(A) J = \tilde{R}(A) \quad (3.14)$$

$$J\Omega = \Omega \quad (3.15)$$

$$\Delta^\dagger = \Delta \quad (3.16)$$

$$J\Delta J = \Delta^{-1} \quad (3.17)$$

$$J\Delta^{1/2} R(A)\Omega + R(A)^\dagger \Omega; \quad (3.18)$$

$$\Delta^{1/2} J\, \tilde{R}(A)\Omega = \tilde{R}(A)^\dagger \Omega. \quad (3.19)$$

Ojima (1981)[15] showed that by introducing the Klein-operator θ; such that $\theta A \theta = \sigma A$ and $\theta^2 = 1$ and by,

accomplishing an adjusted tilde operation (by including a $-\pi/2$ phase factor with all fermion annihilation and creation operators, so that

$$\tilde{\tilde{A}} = A, \tag{3.20}$$

no matter if A is fermionic or bosonic), one can identify

$$\tilde{R}(A) = \begin{cases} i\, J\, R(A) J\theta = -i\, S(A), & \text{fermion} \\ J\, R(A) J = S(A), & \text{boson} \end{cases} \tag{3.21}$$

and the modular operator Δ can be identified as

$$\Delta = \exp(\beta H). \tag{3.22}$$

Hence with equations (3.20), (3.21) and (3.22) we have that equations (3.12) through (3.19) are equivalent to the axioms of equilibrium TFD presented in part II.

In closing this thesis let us note that it has been a long standing problem to obtain a quantum field theoretic formalism which can firstly accommodate thermal degrees of freedom and secondly exhibit irreversibility on a macroscopic level. The discussion in this part has shown that there are methods which put in the temperature parameter by manipulating or forsaking the time parameter and put in irreversibility by hand. TFD approaches these problems by accommodating temperature through the introduction of the tilde field and allows for irreversibility by not requiring a Hermitian Hamiltonian. Since these generalizations can be realized within a quantum field theoretic framework we have a most powerful tool for describing nature.

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Appendix A. Relations Between Thermal Averages

The thermal average $\langle\langle 1|\tilde{a}a|W(t)\rangle\rangle$ for the semi-free field gives us a relation between $\langle\langle 1|a^{\dagger}a|W(t)\rangle\rangle$ and $\langle\langle 1|W(t)\rangle\rangle$ as

$$\langle\langle 1|a^{\dagger}a|W(t)\rangle\rangle = [f^{-1}(t-t_o)-\sigma]^{-1}\langle\langle 1|W(t)\rangle\rangle, \quad (A.1)$$

where we used (2.1), (2.11), (2.13) and thermal state condition (3.9). the relation (3.19) is obtained from (A.1) with (2.3).

The thermal average $\langle\langle 1|W^{\dagger}(t)\tilde{a}a|W(t)\rangle\rangle = \langle\langle W^{\dagger}(t)|\tilde{a}a|W(t)\rangle\rangle$ for the semi-free field gives us a relation between $\langle\langle W^{\dagger}(t)|a^{\dagger}a|W(t)\rangle\rangle$ and $\langle\langle W^{\dagger}(t)|W(t)\rangle\rangle$ as

$$\langle\langle W^{\dagger}(t)|a^{\dagger}a|W(t)\rangle\rangle = [f^{-2}(t-t_o)-\sigma]^{-1}\langle\langle W^{\dagger}(t)|W(t)\rangle\rangle, \quad (A.2)$$

where we used (2.1), (2.11), (2.13) and the thermal state condition (3.9) and its Hermite conjugate. To obtain (5.22), we used (A.2).

Similarly, we can obtain relations between observable thermal averages.